

# Bubbles and Credit Constraints\*

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## Abstract

We provide an infinite-horizon model of a production economy with bubbles, in which firms meet stochastic investment opportunities and face credit constraints. Capital is not only an input for production, but also serves as collateral. We show that bubbles on this reproducible asset may arise, which relax collateral constraints and improve investment efficiency. The collapse of bubbles leads to a recession and a stock market crash. We show that there is a credit policy that can eliminate the bubble on firm assets and can achieve the efficient allocation.

*Keywords:* Bubbles, Collateral Constraints, Credit Policy, Asset Price, Arbitrage, Q Theory, Liquidity, Multiple Equilibria

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# 1 Introduction

Historical evidence has revealed that many countries have experienced large economic fluctuations that may be attributed to asset price bubbles. On the other hand, a number of researchers argue that credit market frictions are important for economic fluctuations (e.g., Bernanke and Gertler (1989), Carlstrom and Fuerst (1997), Kiyotaki and Moore (1997), Bernanke, Gertler and Gilchrist (1999), and Miao and Wang (2010)). In particular, they may amplify and propagate exogenous shocks to the economy. In this paper, we argue that credit market frictions in the form of endogenous credit constraints may create rational bubbles on reproducible assets and the collapse of bubbles leads to a recession.

To formalize this idea, we construct a tractable model in which households are infinitely lived and trade firm stocks. We assume households have linear utility so that the interest rate is equal to the constant subjective discount rate. There is no aggregate uncertainty. A continuum of firms meet stochastic investment opportunities as in Kiyotaki and Moore (2005, 2008) and face credit constraints. We model credit constraints in a way similar to Kiyotaki and Moore (1997). Specifically, durable assets (or capital in our model) are used not only as inputs for production, but also as collateral for loans. Borrowing is limited by the market value of the collateral. Unlike Kiyotaki and Moore (1997) who assume that the market value of the collateral is equal to the fundamental value of the collateralized assets, we allow the market value to contain a bubble component. If both lenders and the credit-constrained borrowers (firms in our model) believe that the collateral values are high possibly because of bubbles, firms want to borrow more and lenders are willing to lend more. Consequently, firms can finance more investment and accumulate more assets for the future production, making their assets indeed more valuable. Because collateral values are equal to the market values of the collateralized assets, the lenders' and the borrowers' beliefs are self-fulfilling and bubbles may sustain in equilibrium. We refer to this equilibrium as the bubbly equilibrium.

Of course, there is another equilibrium in which no one believes in bubbles and hence bubbles do not appear. We call this equilibrium bubbleless equilibrium. We provide explicit conditions to ensure which type of equilibrium can exist. We show that if the collateral constraint is sufficiently tight, then both bubbleless and bubbly equilibria can exist; otherwise, only the bubbleless equilibrium exists. This result is intuitive. If the collateral constraint is too tight, investors have incentives to inflate their asset values to relax the collateral constraint and

bubbles may emerge. If the collateral constraint is too loose, investors can borrow enough to make investment. There is no need for them to create bubbles.

We prove that the bubbly equilibrium has two steady states: one is bubbly and the other is bubbleless. Both steady states are inefficient due to credit constraints. We show that both steady states are local saddle points.<sup>1</sup> The stable manifold is one dimensional for the bubbly steady state, while it is two dimensional for the bubbleless steady state. On the former stable manifold, bubbles persist in the steady state. But on the latter stable manifold, bubbles eventually burst.

As Tirole (1982) and Santos and Woodford (1997) point out, it is hard to generate rational bubbles for economies with infinitely-lived agents. The intuition is the following. A necessary condition for bubbles to exist is that the growth rate of bubbles cannot exceed the growth rate of the economy. Otherwise, investors cannot afford to buy bubbles. In a deterministic economy, bubbles on assets with exogenous payoffs or on intrinsically useless assets must grow at the interest rate by a no-arbitrage argument. Thus, the interest rate cannot exceed the growth rate of the economy. This implies that the present value of aggregate endowments must be infinity. In an overlapping generation economy, this condition implies that the bubbleless equilibrium must be dynamically inefficient (see Tirole (1985)).

In our model, the growth rate of the economy is zero and the interest rate is positive. In addition, the bubbleless equilibrium is dynamically efficient. How to reconcile our result with that in Santos and Woodford (1997) or Tirole (1985)? The key is that bubbles in our model are on reproducible assets with endogenous payoffs. A distinguishing feature of our model is that bubbles on firm assets have real effects and affect the payoffs of these assets. Although a no arbitrage equation for these bubbles still holds in that the rate of return on bubbles is equal to the interest rate, the growth rate of bubbles is not equal to the interest rate. Rather, it is equal to the interest rate minus the “dividend yield.” The dividend yield comes from the fact that bubbles help relax the collateral constraints and allow firms to make more investment. It is equal to the arrival rate of the investment opportunity times the net benefit of new investment (i.e., Tobin’s marginal Q minus 1).

So far, we have only considered deterministic bubbles. Following Blanchard and Watson (1982) and Weil (1989), we construct a third type of equilibrium with stochastic bubbles. In

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<sup>1</sup>Bubbly equilibrium is different from sunspot equilibrium in the indeterminacy literature surveyed by Benhabib and Farmer (1999).

this equilibrium, households believe that there is a positive probability that bubbles burst at each date. When bubbles burst, they cannot reappear again. We show that when the bursting probability is small enough, an equilibrium with stochastic bubbles exists. In contrast to Weil (1989), we show that after the bubble bursts, a recession occurs in that consumption and output fall eventually. In addition, immediately after the bubble bursts, investment falls discontinuously and the stock market crashes in that the stock price falls discontinuously.

What is an appropriate government policy in the wake of the bubble collapse? The inefficiency in our model comes from the firms' credit constraints. The collapse of bubbles tightens these constraints and impair investment efficiency. To overcome this inefficiency, the government may issue public bonds backed by lump-sum taxes. Both households and firms can trade these bonds. They serve as a store of value to households and firms, and also as collateral to firms. Thus, public assets can relax collateral constraints and play the same role as bubbles do. They deliver dividends to firms, but not to households directly. No arbitrage forces these dividends to zero, making Tobin's marginal Q equal to one. This leads to the efficient capital stock. To support the efficient allocation in equilibrium, the government constantly retires public bonds at the interest rate to maintain a constant total bond value and pays the interest payments of these bonds by levying lump-sum taxes. We show that this policy also completely eliminates the bubbles on firm assets.

Some papers in the literature (e.g., Scheinkman and Weiss (1986), Kocherlakota (1992, 1998), Santos and Woodford (1997) and Hellwig and Lorenzoni (2009)) also find that infinite-horizon models with borrowing constraints may generate bubbles. Unlike these papers which study pure exchange economies, our paper analyzes a production economy. As mentioned above, our paper differs from these papers and most papers in the literature in that bubbles in our model are on reproducible assets whose payoffs are affected by bubbles endogenously.<sup>2</sup>

Our paper is closely related to Caballero and Krishnamurthy (2006), Kocherlakota (2009), Wang and Wen (2009), Farhi and Tirole (2010), and Martin and Ventura (2010). Like our paper, these papers contain the idea that bubbles can help relax borrowing constraints and improve investment efficiency. Building on Kiyotaki and Moore (2008), Kocherlakota (2009) studies an economy with infinitely-lived entrepreneurs. Entrepreneurs meet stochastic investment opportunities and are subject to collateral constraints. Land is used as the collateral.

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<sup>2</sup>See Scheinkman and Xiong (2003) for a model of irrational bubbles. See Brunnermeier (2009) for a survey of models of bubbles.

Unlike Kiyotaki and Moore (1997) or our paper, Kocherlakota (2009) assumes that land is intrinsically useless (i.e. it has no rents or dividends) and cannot be used as an input for production. Wang and Wen (2009) provide a model similar to that in Kocherlakota (2009). They study asset price volatility and bubbles that may grow on assets with exogenous rents. They also assume that these assets cannot be used as an input for production.

Building on Diamond (1965) and Tirole (1985), Caballero and Krishnamurthy (2006), Farhi and Tirole (2010), and Martin and Ventura (2010) study bubbles in overlapping-generation models with credit constraints. Caballero and Krishnamurthy (2006) show that stochastic bubbles are beneficial because they provide domestic stores of value and thereby reduce capital outflows while increasing investment. But they come at a cost, as they expose the country to bubble crashes and capital flow reversals. Farhi and Tirole (2010) assume that entrepreneurs may use bubbles and outside liquidity to relax the credit constraints. They study the interplay between inside and outside liquidity. Martin and Ventura (2010) use a model with bubbles to shed light on the current financial crisis.

Our discussion of credit policy is related to Caballero and Krishnamurthy (2006) and Kocherlakota (2009). As in their studies, government bonds can serve as collateral to relax credit constraints in my model. Unlike their proposed policies, my proposed policy requires that government bonds be backed by lump-sum taxes and it can make the economy achieve the efficient allocation.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 derives the equilibrium system. Section 4 analyzes the bubbleless equilibrium, while Section 5 analyzes the bubbly equilibrium. Section 6 studies stochastic bubbles. Section 7 introduces public assets and studies government credit policy. Section 8 concludes. An appendix contains technical proofs.

## 2 The Base Model

We consider an infinite-horizon economy. There is no aggregate uncertainty. Time is denoted by  $t = 0, dt, 2dt, 3dt, \dots$ . The length of a time period is  $dt$ . For analytical convenience, we shall take the limit of this discrete-time economy as  $dt$  goes to zero when characterizing equilibrium dynamics. Instead of presenting a continuous-time model directly, we start with the model in discrete time in order to make the intuition transparent.

## 2.1 Households

There is a continuum of identical households with a unit mass. Each household is risk neutral and derives utility from a consumption stream  $\{C_t\}$  according to the following utility function:

$$\sum_{t \in \{0, dt, 2dt, \dots\}} e^{-rt} C_t dt,$$

where  $r$  is the subjective rate of time preference.<sup>3</sup> Households supply labor inelastically. The labor supply is normalized to one. Households trade firm stocks and riskfree household bonds. The net supply of household bonds is zero and the net supply of any stock is one. Because there is no aggregate uncertainty,  $r$  is equal to the riskfree rate (or interest rate) and also equal to the rate of the return for each stock.

## 2.2 Firms

There is a continuum of firms with a unit mass. Firms are indexed by  $j \in [0, 1]$ . Each firm  $j$  combines labor  $N_t^j$  and capital  $K_t^j$  to produce output according to the following Cobb-Douglas production function:

$$Y_t^j = (K_t^j)^\alpha (N_t^j)^{1-\alpha}, \quad \alpha \in (0, 1).$$

After solving the static labor choice problem, we obtain the operating profits

$$R_t K_t^j = \max_{N_t^j} (K_t^j)^\alpha (N_t^j)^{1-\alpha} - w_t N_t^j, \quad (1)$$

where  $w_t$  is the wage rate and

$$R_t = \alpha \left( \frac{w_t}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}}. \quad (2)$$

We will show later that  $R_t$  is equal to the marginal product of capital or the rental rate of capital.

Each firm  $j$  meets an opportunity to make investment in capital with probability  $\pi dt$  in period  $t$ . With probability  $1 - \pi dt$ , no investment opportunity arrives. Thus, capital evolves according to:

$$K_{t+dt}^j = \begin{cases} (1 - \delta dt) K_t^j + I_t^j & \text{with probability } \pi dt \\ (1 - \delta dt) K_t^j & \text{with probability } 1 - \pi dt \end{cases}, \quad (3)$$

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<sup>3</sup>Introducing a general concave utility function allows to endogenize interest rate, but it makes analysis more complex. It will not change our key insights.

where  $\delta > 0$  is the depreciation rate of capital and  $I_t^j$  is the investment level. Assume that the arrival of the investment opportunity is independent across firms and over time.

Let the expected firm value (or stock value) be  $V_t(K_t^j)$ . It satisfies the following Bellman equation:

$$V_t(K_t^j) = \max_{I_t^j} R_t K_t^j dt - \pi I_t^j dt + e^{-rdt} V_{t+dt} ((1 - \delta dt) K_t^j + I_t^j) \pi dt \quad (4)$$

$$+ e^{-rdt} V_{t+dt} ((1 - \delta dt) K_t^j) (1 - \pi dt),$$

subject to some constraints on investment to be specified next. As will be shown in Section 3, the optimization problem in (4) is not well defined if there is no constraint on investment given our assumption of the constant returns to scale technology. Thus, we impose some upper bound and lower bound on investment.<sup>4</sup> For the lower bound, we assume that investment is irreversible in that  $I_t^j \geq 0$ . It turns out this constraint will never bind in our analysis below. For the upper bound, we assume that investment is financed by internal funds and external borrowing. We also assume that external equity is so costly that all firms do not raise new equity to finance investment.<sup>5</sup> Our model applies better to emerging economies with less developed equity markets.

We now write the investment constraint as:

$$0 \leq I_t^j \leq R_t K_t^j + L_t^j, \quad (5)$$

where  $R_t K_t^j$  represents internal funds and  $L_t^j$  represents loans from financial intermediaries. To reduce the number of state variables and keep the model tractable, we consider intratemporal loans as in Jermann and Quadrini (2010).<sup>6</sup> These loans are taken at the beginning of the period and repaid at the end of the period. They do not have interests. In future research, it would be interesting to incorporate intertemporal corporate bonds with interest payments.

The key assumption of our model is that loans are subject to collateral constraints, as in Kiyotaki and Moore (1997). Firm  $j$  pledges a fraction  $\xi \in (0, 1]$  of its assets (capital stock)  $K_t^j$

<sup>4</sup>Alternatively, one may impose convex adjustment costs of investment.

<sup>5</sup>This assumption reflects the fact that external equity financing is more costly than debt financing. Bernanke et al. (1999), Carlstrom and Fuerst (1997), and Kiyotaki and Moore (1997) make the same assumptions. We can relax this assumption by allowing firms to raise a limited amount of new equity so that we can rewrite (5) as

$$I_t^j \leq R_t K_t^j + a K_t^j + L_t^j,$$

where  $a K_t^j$  represents the upper bound of new equity. In this case, our analysis and insights still go through with small modification.

<sup>6</sup>Jermann and Quadrini (2010) also consider intertemporal corporate bonds.

at the beginning of period  $t$  as the collateral. The parameter  $\xi$  may represent the tightness of the collateral constraint or the extent of financial market imperfections. It is the key parameter for our analysis below. In the end of period  $t$ , the market value of the collateral is equal to  $e^{-rdt}V_{t+dt}(\xi K_t^j)$ . This is the discounted expected market value of the firm if firm  $j$  owns capital stock  $\xi K_t^j$  at the beginning of period  $t + dt$  and faces the same investment constraint and collateral constraint in the future. The amount of loans  $L_t^j$  cannot exceed this collateral value. Otherwise, the firm would choose to default on debt and lose the collateral value. Thus, we impose the following collateral constraint:

$$L_t^j \leq e^{-rdt}V_{t+dt}(\xi K_t^j). \quad (6)$$

In the continuous time limit, this constraint becomes<sup>7</sup>

$$L_t^j \leq V_t(\xi K_t^j). \quad (7)$$

Note that our modelling of collateral constraint is different from Kiyotaki and Moore (1997). In their model, borrowing is limited by the market value of the assets, and this market value is assumed to be the fundamental value. We may write the Kiyotaki-Moore-type collateral constraint in our model framework as:

$$L_t^j \leq \xi Q_t K_t^j, \quad (8)$$

where  $Q_t$  is the capital price. Here,  $\xi Q_t K_t^j$  is the fundamental market value of the collateralized assets  $\xi K_t^j$ . Constraint (8) implies that firms cannot use bubbles to relax collateral constraints. In Section 5, we shall argue that this type of collateral constraint will rule out bubbles.

### 2.3 Competitive Equilibrium

Let  $K_t = \int_0^1 K_t^j dj$ ,  $I_t = \int_0^1 I_t^j dj$ ,  $N_t = \int_0^1 N_t^j dj$ , and  $Y_t = \int_0^1 Y_t^j dj$  be the aggregate capital stock, the aggregate investment, the aggregate labor demand, and aggregate output. Then a competitive equilibrium is defined as sequences of  $\{Y_t\}$ ,  $\{C_t\}$ ,  $\{K_t\}$ ,  $\{I_t\}$ ,  $\{N_t\}$ ,  $\{w_t\}$ ,  $\{R_t\}$ ,

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<sup>7</sup>Alternatively, we may assume that the collateral constraint is given by:

$$L_t^j \leq \xi V_t(K_t^j).$$

In this case, we interpret  $\xi$  as the liquidity discount of firm value in the secondary market. This modelling does not change our analysis significantly.

$\{V_t(K_t^j)\}$ ,  $\{I_t^j\}$ ,  $\{K_t^j\}$ ,  $\{N_t^j\}$  and  $\{L_t^j\}$  such that households and firms optimize and markets clear in that:

$$\begin{aligned} N_t &= 1, \\ C_t + \pi I_t &= Y_t, \\ K_{t+dt} &= (1 - \delta dt) K_t + I_t \pi dt. \end{aligned}$$

### 3 Equilibrium System

We first solve an individual firm's optimization problem (4) subject to (3), (5), and (6) when the wage rate  $w_t$  or the rental rate  $R_t$  in (2) is taken as given. We conjecture that firm value takes the following form:

$$V_t(K_t^j) = v_t K_t^j + b_t, \quad (9)$$

where  $v_t$  and  $b_t$  are to be determined variables that depend on aggregate states only. We may interpret  $v_t K_t^j$  as the fundamental value and  $b_t$  as bubbles if  $b_t \neq 0$ .<sup>8</sup> The fundamental value is related to the firm's assets  $K_t^j$  and the bubbles are unrelated to them. Note that  $b_t = 0$  is also a possible solution. Let  $Q_t$  be the Lagrange multiplier associated with the constraint (3) if the investment opportunity arrives. It represents the shadow price of capital or marginal  $Q$ . The following result characterizes firm  $j$ 's optimization problem:

**Proposition 1** *Suppose  $Q_t > 1$  and let  $w_t$  be given. Then the optimal investment level when the investment opportunity arrives is given by:*

$$I_t^j = R_t K_t^j + \xi Q_t K_t^j + B_t, \quad (10)$$

where  $R_t$  is given by (2) and

$$B_t = e^{-rdt} b_{t+dt}, \quad (11)$$

$$Q_t = e^{-rdt} v_{t+dt}. \quad (12)$$

In addition,

$$v_t = R_t dt + (1 - \delta dt) Q_t + (Q_t - 1) (R_t + \xi Q_t) \pi dt, \quad (13)$$

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<sup>8</sup>In the literature on investment under uncertainty, firm value consists of the value of assets in place plus the value of the growth option in the presence of an irreversible investment project with lump-sum costs. Our model does not have this feature. In our model, firms make continuous capital adjustments and the term  $b_t$  comes from the endogenous credit constraints rather than growth options.

$$b_t = B_t + (Q_t - 1) B_t \pi dt. \quad (14)$$

and the transversality condition holds:

$$\lim_{T \rightarrow \infty} e^{-rTdt} Q_T K_{T+dt}^j = 0, \quad \lim_{T \rightarrow \infty} e^{-rTdt} b_T = 0.$$

The intuition for this proposition is the following. When the investment opportunity arrives, an additional unit of investment costs the firm one unit of the consumption good, but generates an additional value of  $Q_t$ , where  $Q_t$  satisfies (12). This equation and equation (9) reveal that

$$Q_t = e^{-rdt} \frac{\partial V_{t+dt}(K_{t+dt})}{\partial K_{t+dt}}.$$

Thus,  $Q_t$  represents the marginal value of the firm following a unit increase in capital at time  $t + dt$  in time- $t$  dollars, i.e., Tobin's marginal  $Q$ . If  $Q_t > 1$ , the firm will make the maximal possible level of investment. If  $Q_t = 1$ , the investment level is indeterminate. If  $Q_t < 1$ , the firm will make the minimal possible level of investment. This investment choice is similar to Tobin's  $Q$  theory (Tobin (1969) and Hayashi (1982)). In what follows, we impose assumptions to ensure  $Q_t > 1$  at least in the neighborhood of the steady state equilibrium. We thus obtain the investment rule given in (10). Substituting this rule and equation (9) into the Bellman equation (4) and matching coefficients, we obtain equations (13) and (14).

Although our model features constant returns to scale, marginal  $Q$  is not equal to average  $Q$  in the presence of bubbles, because average  $Q$  is equal to

$$\frac{e^{-rdt} V_{t+dt}(K_{t+dt})}{K_{t+dt}} = Q_t + \frac{B_t}{K_{t+dt}}, \text{ for } B_t \neq 0.$$

Thus, the existence of stock price bubbles invalidates Hayashi's (1982) result. In the empirical investment literature, researchers typically use average  $Q$  to replace marginal  $Q$  under the constant returns to scale assumption because marginal  $Q$  is not observable. Our analysis demonstrates that the existence of collateral constraints implies that stock prices may contain a bubble component that makes marginal  $Q$  not equal to average  $Q$ .

Next, we aggregate individual firm's decision rules and impose market-clearing conditions. We then characterize a competitive equilibrium by a system of nonlinear difference equations:

**Proposition 2** *Suppose  $Q_t > 1$ . Then the equilibrium sequences  $(B_t, Q_t, K_t)$ , for  $t = 0, dt, 2dt, \dots$ , satisfy the following system of nonlinear difference equations:*

$$B_t = e^{-rdt} B_{t+dt} [1 + \pi(Q_{t+dt} - 1)dt], \quad (15)$$

$$Q_t = e^{-r dt} [R_{t+dt} dt + (1 - \delta dt) Q_{t+dt} + (R_{t+dt} + \xi Q_{t+dt}) (Q_{t+dt} - 1) \pi dt], \quad (16)$$

$$K_{t+dt} = (1 - \delta dt) K_t + \pi (R_t K_t + \xi Q_t K_t + B_t) dt, \quad K_0 \text{ given}, \quad (17)$$

and the transversality condition:

$$\lim_{T \rightarrow \infty} e^{-rT dt} Q_T K_{T+dt} = 0, \quad \lim_{T \rightarrow \infty} e^{-rT dt} B_T = 0,$$

where  $R_t = \alpha K_t^{\alpha-1}$ .

When  $dt = 1$ , the above system reduces to the usual discrete-time characterization of equilibrium. However, this system is not convenient for analytically characterizing local dynamics. We may solve this system numerically by assigning parameter values. Instead of pursuing this route, we use analytical methods in the continuous-time limit as  $dt$  goes to zero. To compute the limit, we use the heuristic rule  $dX_t = X_{t+dt} - X_t$  for any variable  $X_t$ . We also use the notation  $\dot{X}_t = dX_t/dt$ . We obtain the following:

**Proposition 3** *Suppose  $Q_t > 1$ . Then in the continuous-time limit as  $dt \rightarrow 0$ , the equilibrium dynamics  $(B_t, Q_t, K_t)$  satisfy the following system of differential equations:*

$$\dot{B}_t = r B_t - B_t \pi (Q_t - 1), \quad (18)$$

$$\dot{Q}_t = (r + \delta) Q_t - R_t - \pi (R_t + \xi Q_t) (Q_t - 1), \quad (19)$$

$$\dot{K}_t = -\delta K_t + \pi (R_t K_t + \xi Q_t K_t + B_t), \quad K_0 \text{ given}, \quad (20)$$

and the transversality condition:

$$\lim_{T \rightarrow \infty} e^{-rT} Q_T K_T = 0, \quad \lim_{T \rightarrow \infty} e^{-rT} B_T = 0,$$

where  $R_t = \alpha K_t^{\alpha-1}$ . In addition,  $Q_t = v_t$  and  $B_t = b_t$  so that the market value of firm  $j$  is given by  $V_t(K_t^j) = Q_t K_t^j + B_t$ .

After obtaining the solution for  $(B_t, Q_t, K_t)$ , we can derive the equilibrium wage rate  $w_t = (1 - \alpha) K_t^\alpha$ , the rental rate  $R_t = \alpha K_t^{\alpha-1}$ , aggregate output  $Y_t = K_t^\alpha$ , aggregate investment

$$I_t = R_t K_t + \xi Q_t K_t + B_t, \quad (21)$$

and aggregate consumption  $C_t = Y_t - \pi I_t$ . Clearly there are two types of equilibrium. The first type is bubbleless, for which  $B_t = 0$  for all  $t$ . In this case, the market value of firm  $j$  is equal

to its fundamental value in that  $V_t(K_t^j) = Q_t K_t^j$ . The second type is bubbly, for which  $B_t \neq 0$  for some  $t$ . We assume that assets can be freely disposed of so that the bubbles  $B_t$  cannot be negative. In this case, firm value contains a bubble component in that  $V_t(K_t^j) = Q_t K_t^j + B_t$  with  $B_t > 0$ . We next study these two types of equilibrium.

## 4 Bubbleless Equilibrium

In a bubbleless equilibrium,  $B_t = 0$  for all  $t$ . Equation (18) becomes an identity. We only need to focus on  $(Q_t, K_t)$  determined by the differential equations (19) and (20) in which  $B_t = 0$  for all  $t$ . In the continuous time limit,  $v_t = Q_t$

We first analyze the steady state. In the steady state, all aggregate variables are constant over time so that  $\dot{Q}_t = \dot{K}_t = 0$ . We use  $X$  to denote the steady state value of any variable  $X_t$ . By (19) and (20), we obtain the following steady-state equations:

$$0 = (r + \delta)Q - R - \pi(R + \xi Q)(Q - 1), \quad (22)$$

$$0 = -\delta K + \pi(RK + \xi QK). \quad (23)$$

We use a variable with an asterisk to denote its value in the bubbleless equilibrium. Solving equations (22)-(23) yields:

**Proposition 4** (i) If

$$\xi \geq \frac{\delta(1 - \pi)}{\pi} - r, \quad (24)$$

then there exists a unique bubbleless steady state equilibrium with  $Q_t^* = Q_E \equiv 1$  and  $K_t^* = K_E$ , where  $K_E$  is the efficient capital stock satisfying  $\alpha(K_E)^{\alpha-1} = r + \delta$ .

(ii) If

$$0 < \xi < \frac{\delta(1 - \pi)}{\pi} - r, \quad (25)$$

then there exists a unique bubbleless steady-state equilibrium with

$$Q^* = \frac{\delta(1 - \pi)}{\pi} \frac{1}{r + \xi} > 1, \quad (26)$$

$$\alpha(K^*)^{\alpha-1} = \frac{\delta(1 - \pi)}{\pi} \frac{r}{r + \xi} + \delta. \quad (27)$$

In addition,  $K^* < K_E$ .

Assumption (24) says that if firms pledge sufficient assets as the collateral, then the collateral constraints will not bind in equilibrium. The competitive equilibrium allocation is the same as the efficient allocation. The efficient allocation is achieved by solving a social planner's problem in which the social planner maximizes the representative household's utility subject to the resource constraint only. Note that we assume that the social planner also faces stochastic investment opportunities, like firms in a competitive equilibrium. Thus, one may view our definition of the efficient allocation as the constrained efficient allocation. Unlike firms in a competitive equilibrium, the social planner is not subject to collateral constraints.

Assumption (25) says that if firms do not pledge sufficient assets as the collateral, then the collateral constraints will be sufficiently tight so that firms are credit constrained in the neighborhood of the steady-state equilibrium in which  $Q^* > 1$ . We can then apply Proposition 3 in this neighborhood. Proposition 4 also shows that the steady-state capital stock for the bubbleless competitive equilibrium is less than the efficient steady-state capital stock. This reflects the fact that not enough resources are transferred from savers to investors due to the collateral constraints.

Note that for (25) to hold, the arrival rate  $\pi$  of the investment opportunity must be sufficiently small, holding everything else constant. The intuition is that if  $\pi$  is too high, then too many firms will have investment opportunities so that the accumulated aggregate capital stock will be large, thereby lowering the capital price  $Q$  to the efficient level 1. In this case, the collateral constraints will not bind and the economy will reach the first best. Thus, condition (25) also requires that technological constraints at the firm level must be sufficiently tight.

Next, we study the stability of the steady state and the dynamics of the equilibrium system. We use the phase diagram in Figure 1 to describe the two dimensional dynamic system for  $(Q_t, K_t)$ . It is straightforward to show that the  $\dot{K}_t = 0$  locus is upward sloping. Above this line,  $\dot{K}_t < 0$ , and below this line  $\dot{K}_t > 0$ . Turn to the  $\dot{Q}_t = 0$  locus. One can verify that on the  $\dot{Q}_t = 0$  locus,  $dK/dQ|_{Q \rightarrow 1} < 0$  and  $dK/dQ|_{Q \rightarrow \infty} > 0$ . But for general  $Q > 1$ , we cannot determine the sign of  $dK/dQ$ . Above the  $\dot{Q}_t = 0$  line,  $\dot{Q}_t > 0$ , and below the  $\dot{Q}_t = 0$  line,  $\dot{Q}_t < 0$ . In addition, the  $\dot{Q}_t = 0$  line and the  $\dot{K}_t = 0$  line have only one crossing point at the steady state  $(Q^*, K^*)$ . The slope along the  $\dot{K}_t = 0$  line is always larger than that along the  $\dot{Q}_t = 0$  line. For  $Q < Q^*$ , the  $\dot{Q}_t = 0$  line is above the  $\dot{K}_t = 0$  line. For  $Q > Q^*$ , the opposite is true. In summary, two cases may happen as illustrated in Figure 1. For both cases, there is a unique saddle path such that for any given initial value  $K_0$ , when  $Q_0$  is on the saddle path,

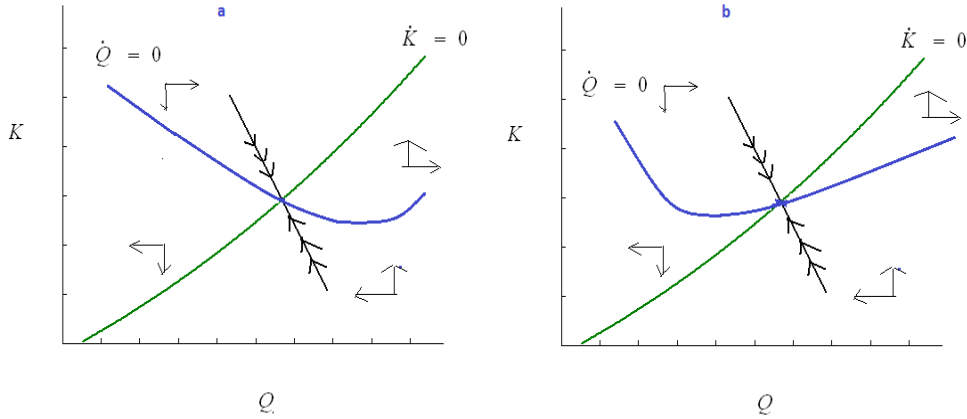


Figure 1: Phase diagram for the dynamics of the bubbleless equilibrium.

the economy approaches the long-run steady state.

## 5 Bubbly Equilibrium

In this section, we study bubbly equilibrium in which  $B_t > 0$  for some  $t$ . We shall analyze the dynamic system for  $(B_t, Q_t, K_t)$  given in (18)-(20). Before we conduct a formal analysis later, we first discuss the intuition for why bubbles can exist in our model. The key is to understand equation (18), rewritten as:

$$\frac{\dot{B}_t}{B_t} + \pi(Q_t - 1) = r, \text{ for } B_t \neq 0. \quad (28)$$

The first term on the left-hand side is the rate of capital gains of bubbles. The second term represents “dividend yields”, as we will explain below. Thus, equation (18) or (28) reflects a no-arbitrage relation in that the rate of return on bubbles must be equal to the interest rate. A similar relation also appears in the literature on rational bubbles, e.g., Blanchard and Watson (1982), Tirole (1985), Weil (1987, 1993), and Farhi and Tirole (2010). This literature typically studies bubbles on zero-payoff assets or unproductive assets with exogenously given payoffs. In this case, the second term on the left-hand side of (28) vanishes and bubbles grow at the rate of interest. If we adopt collateral constraint (8) as in Kiyotaki and Moore (1997), then we can also show that bubbles grow at the rate of interest. In an infinite-horizon economy,

the transversality condition rules out these bubbles. In an overlapping generation economy, for bubbles to exist, the interest rate must be less than the growth rate of the economy in the bubbleless equilibrium. This means that the bubbleless equilibrium must be dynamically inefficient (see Tirole (1985)).

Unlike this literature, bubbles in our model are on reproducible real assets and also influence their fundamentals (or dividends). Specifically, each unit of the bubble raises the collateral value by one unit and hence allows the firm to borrow an additional unit. The firm then makes one more unit of investment when investment opportunity arrives. This unit of investment raises firm value by  $Q_t$ . Subtracting one unit of costs, we then deduce that the second term on the left-hand side of (28) represents the net increase in firm value for each unit of bubbles. This is why we call this term dividend yields. Dividend payouts make the growth rate of bubbles less than the interest rate. Thus, the transversality condition cannot rule out bubbles in our model. We can also show that the bubbleless equilibrium is dynamically efficient in our model. Specifically, the golden rule capital stock is given by  $K_{GR} = (\delta/\alpha)^{\frac{1}{\alpha-1}}$ . One can verify that  $K^* < K_{GR}$ . Thus, one cannot use the condition for the overlapping generation economies in Tirole (1985) to ensure the existence of bubbles. Below we will give new conditions to ensure the existence of bubbles in our model.

## 5.1 Steady State

We first study the existence of a bubbly steady state in which  $B > 0$ . We use a variable with a subscript  $b$  to denote this variable's bubbly steady state value. By Proposition 3,  $(B, Q_b, K_b)$  satisfies equations (22) and

$$0 = rB - B\pi(Q - 1), \quad (29)$$

$$0 = -\delta K + [RK + \xi QK + B]\pi. \quad (30)$$

Using these equations, we can derive:

**Proposition 5** *There exists a bubbly steady state satisfying*

$$\frac{B}{K_b} = \frac{\delta}{\pi} - \frac{r + \delta + \xi r + \pi}{1 + r} \frac{r + \pi}{\pi} > 0, \quad (31)$$

$$Q_b = \frac{r}{\pi} + 1 > 1, \quad (32)$$

$$\alpha (K_b)^{\alpha-1} = \frac{(1 - \xi)r + \delta}{1 + r} \left( \frac{r}{\pi} + 1 \right), \quad (33)$$

if and only if the following condition holds:

$$0 < \xi < \frac{\delta(1-\pi)}{r+\pi} - r. \quad (34)$$

In addition, (i)  $Q_b < Q^*$ , (ii)  $K_{GR} > K_E > K_b > K^*$ , and (iii) the bubble-asset ratio  $B/K_b$  decreases with  $\xi$ .

From equations (22), (29) and (30), we can immediately derive (31)-(33). We can then immediately see that condition (34) is equivalent to  $B/K_b > 0$ . This condition reveals that bubbles occur when  $\xi$  is sufficiently small or the collateral constraint is sufficiently tight. The intuition is the following. When the collateral constraint is too tight, firms prefer to overvalue their assets in order to raise their collateral value. In this way, they can borrow more and invest more. As a result, bubbles may emerge. If the collateral constraint is not tight enough, firms can borrow sufficient funds to finance investment. They have no incentive to create a bubble.

Note that condition (34) implies condition (25). Thus, if condition (34) holds, then there exist two steady state equilibria: one is bubbleless and the other is bubbly. The bubbleless steady state is analyzed in Proposition 4. Propositions 5 and 4 reveal that the steady-state capital price is lower in the bubbly equilibrium than in the bubbleless equilibrium, i.e.,  $Q_b < Q^*$ . The intuition is as follows. In a bubbleless or a bubbly steady state, the investment rate must be equal to the rate of capital depreciation such that the capital stock is constant over time (see equations (23) and (30)). Bubbles relax collateral constraints and induce firms to make more investment, compared to the case without bubbles. To maintain the same steady-state investment rate, the capital price in the bubbly steady state must be lower than that in the bubbleless steady state.

Do bubbles crowd out capital in the steady state? In Tirole's (1985) overlapping generation model, households may use part of savings to buy bubble assets instead of accumulating capital. Thus, bubbles crowd out capital in the steady state. In our model, bubbles are on reproducible assets. If the capital price is the same for both bubbly and bubbleless steady states, then bubbles induce firms to invest more and hence to accumulate more capital stock. However, there is a general equilibrium price feedback effect as discussed earlier. The lower capital price in the bubbly steady state discourages firms to accumulate more capital stock. The net effect is that bubbles lead to higher capital accumulation, unlike Tirole's (1985) result. However, bubbles still do not lead to the efficient allocation. The capital stock in the bubbly steady state is still lower than that in the efficient allocation.

How does the tightness of collateral constraint affect the size of bubbles. Proposition 5 shows that a tighter collateral constraint (i.e., a smaller  $\xi$ ) leads to a larger size of bubbles relative to capital. This is intuitive. Facing a tighter collateral constraint, firms have more incentives to generate larger bubbles to finance investment.

## 5.2 Dynamics

Now, we study the stability of the two steady states and the local dynamics around these steady states. Since the equilibrium system (18)-(20) is three dimensional, we cannot use the phase diagram to analyze its stability. We thus consider a linearized system and obtain the following:

**Proposition 6** *Suppose condition (34) holds. Then both the bubbly steady state  $(B, Q_b, K_b)$  and the bubbleless steady state  $(0, Q^*, K^*)$  are local saddle points for the nonlinear system (18)-(20).*

More formally, in the appendix, we prove that for the nonlinear system (18)-(20), there is a neighborhood  $\mathcal{N} \subset \mathbb{R}_+^3$  of the bubbly steady state  $(B, Q_b, K_b)$  and a continuously differentiable function  $\phi : \mathcal{N} \rightarrow \mathbb{R}^2$  such that given any  $K_0$  there exists a unique solution  $(B_0, Q_0)$  to the equation  $\phi(B_0, Q_0, K_0) = 0$  with  $(B_0, Q_0, K_0) \in \mathcal{N}$ , and  $(B_t, Q_t, K_t)$  converges to  $(B, Q_b, K_b)$  starting at  $(B_0, Q_0, K_0)$  as  $t$  approaches infinity. The set of points  $(B, Q, K)$  satisfying the equation  $\phi(B, Q, K) = 0$  is a one dimensional stable manifold of the system. If the initial value  $(B_0, Q_0, K_0)$  is on the stable manifold, then the solution to the nonlinear system (18)-(20) is also on the stable manifold and converges to  $(B, Q_b, K_b)$  as  $t$  approaches infinity.

Although the bubbleless steady state  $(0, Q^*, K^*)$  is also a local saddle point, the local dynamics around this steady state are different. In the appendix, we prove that the stable manifold for the bubbleless steady state is two dimensional. Formally, there is a neighborhood  $\mathcal{N}^* \subset \mathbb{R}_+^3$  of  $(0, Q^*, K^*)$  and a continuously differentiable function  $\phi^* : \mathcal{N}^* \rightarrow \mathbb{R}$  such that given any  $(B_0, K_0)$  there exists a unique solution  $Q_0$  to the equation  $\phi^*(B_0, Q_0, K_0) = 0$  with  $(B_0, Q_0, K_0) \in \mathcal{N}^*$ , and  $(B_t, Q_t, K_t)$  converges to  $(0, Q^*, K^*)$  starting at  $(B_0, Q_0, K_0)$  as  $t$  approaches infinity. Intuitively, along the two dimensional stable manifold, the bubbly equilibrium is asymptotically bubbleless in that bubbles will burst eventually.

## 6 Stochastic Bubbles

So far, we have focused on deterministic bubbles. Following Blanchard and Watson (1982) and Weil (1987), we now study stochastic bubbles. Consider a discrete-time economy described as in Section 2. Suppose a bubble exists initially,  $B_0 > 0$ . In each time interval between  $t$  and  $t + dt$ , there is a constant probability  $\theta dt$  that the bubble bursts,  $B_{t+dt} = 0$ . Once it bursts, it will never be valued again in the future so that  $B_\tau = 0$  for all  $\tau \geq t + dt$ . With the remaining probability  $1 - \theta dt$ , the bubble persists so that  $B_{t+dt} > 0$ . Later, we will take the continuous time limits as  $dt \rightarrow 0$ .

First, we consider the case in which the bubble has collapsed. This corresponds to the bubbleless equilibrium studied in Section 4. We use a variable with an asterisk (except for  $K_t$ ) to denote its value in the bubbleless equilibrium. In particular,  $V_t^*(K_t^j)$  denotes firm  $j$ 's value function. In the continuous-time limit,  $(Q_t^*, K_t)$  satisfies the equilibrium system (19) and (20) with  $B_t = 0$ . We may express the solution for  $Q_t^*$  in a feedback form in that  $Q_t^* = g(K_t)$  for some function  $g$ .

Next, we consider the case in which the bubble has not bursted. We write firm  $j$ 's dynamic programming problem as follows:

$$\begin{aligned}
 V_t(K_t^j) = \max & R_t K_t^j dt - \pi I_t^j dt \\
 & + e^{-rdt} (1 - \theta dt) V_{t+dt}((1 - \delta dt)K_t^j + I_t^j)\pi dt \\
 & + e^{-rdt} (1 - \theta dt) V_{t+dt}((1 - \delta dt)K_t^j) (1 - \pi dt) \\
 & + e^{-rdt} \theta dt V_{t+dt}^*((1 - \delta dt)K_t^j + I_t^j)\pi dt \\
 & + e^{-rdt} \theta dt V_{t+dt}^*((1 - \delta dt)K_t^j) (1 - \pi dt)
 \end{aligned} \tag{35}$$

subject to (5) and

$$L_t^j \leq e^{-rdt} V_{t+dt}(\xi K_t^j) (1 - \theta dt) + e^{-rdt} V_{t+dt}^*(\xi K_t^j) \theta dt. \tag{36}$$

We conjecture that the value function takes the form:

$$V_t(K_t^j) = v_t K_t^j + b_t, \tag{37}$$

where  $v_t$  and  $b_t$  are to be determined variables that are independent of  $K_t^j$ . As we have shown in Section 4, when the bubble bursts, the value function satisfies:

$$V_t^*(K_t^j) = v_t^* K_t^j. \tag{38}$$

Substituting the above two equations into (35) and simplifying, we rewrite the firm's dynamic programming problem as:

$$v_t K_t^j + b_t = \max R_t K_t^j dt - \pi I_t^j dt + Q_t(1 - \delta dt)K_t^j + Q_t \pi I_t^j dt + B_t, \quad (39)$$

subject to

$$0 \leq I_t^j \leq R_t K_t^j + Q_t \xi K_t^j + B_t, \quad (40)$$

where we define  $Q_t^* = e^{-r dt} v_{t+dt}^*$ ,

$$Q_t = e^{-r dt} [(1 - \theta dt)v_{t+dt} + \theta v_{t+dt}^*], \quad (41)$$

$$B_t = e^{-r dt} (1 - \theta dt)b_{t+dt}. \quad (42)$$

Suppose  $Q_t > 1$ . Then optimal investment achieves the upper bound in (40). Substituting this investment level into equation (39) and matching coefficients on the two sides of this equation, we obtain:

$$v_t = R_t dt + Q_t(1 - \delta dt) + \pi(Q_t - 1)(R_t + Q_t \xi) dt, \quad (43)$$

$$b_t = B_t + \pi(Q_t - 1)B_t dt. \quad (44)$$

As in Section 3, we conduct aggregation to obtain the discrete-time equilibrium system. We then take the continuous-time limits as  $dt \rightarrow 0$  to obtain the following:

**Proposition 7** *Suppose  $Q_t > 1$ . When the bubble does not burst, the equilibrium with stochastic bubbles  $(B_t, Q_t, K_t)$  satisfies the following system of differential equations:*

$$\dot{B}_t = (r + \theta)B_t - \pi(Q_t - 1)B_t, \quad (45)$$

$$\dot{Q}_t = (r + \delta + \theta)Q_t - \theta Q_t^* - R_t - \pi(Q_t - 1)(R_t + \xi Q_t), \quad (46)$$

and (20), where  $R_t = \alpha K_t^{\alpha-1}$  and  $Q_t^* = g(K_t)$  is the capital price after the bubble bursts.

Equation (45) reveals that the rate of return on bubbles is equal to  $r + \theta$ , which is higher than the interest rate. This reflects risk premium because the stochastic bubble is risky. In general, it is hard to characterize the equilibrium with stochastic bubbles. In order to transparently illustrate the adverse impact of bubble bursting on the economy, we shall consider a simple type of equilibrium. Following Weil (1987) and Kocherlakota (2009), we study a stationary equilibrium with stochastic bubbles that has the following properties: The capital stock is

constant at the value  $K_s$  over time before the bubble collapses. It continuously moves to the bubbleless steady state value  $K^*$  after the bubble collapses. The bubble is also constant at the value  $B_s > 0$  before the bubble collapses. It jumps to zero and then stays at this value after the bubble collapses. The capital price is constant at the value  $Q_s$  before the bubble collapses. It jumps to the value  $g(K)$  after it collapses and then converges to the bubbleless steady-state value  $Q^*$  given in equation (26).

Our objective is to show the existence of  $(B_s, Q_s, K_s)$ . By (45), we can show that

$$Q_s = \frac{r + \theta}{\pi} + 1. \quad (47)$$

Since  $Q_s > 1$ , we can apply Proposition 7 in some neighborhood of  $Q_s$ . Equation (46) implies that

$$0 = (r + \delta + \theta)Q_s - \theta g(K) - R - \pi(Q_s - 1)(R + \xi Q_s), \quad (48)$$

where  $R = \alpha K^{\alpha-1}$ . The solution to this equation gives  $K_s$ . Once we obtain  $K_s$  and  $Q_s$ , we use equation (30) to determine  $B_s$ .

The difficult part is to solve for  $K_s$ . In doing so, we define  $\theta^*$  such that

$$\frac{r + \theta^*}{\pi} + 1 = \frac{\delta(1 - \pi)}{\pi} \frac{1}{r + \xi} = Q^*. \quad (49)$$

That is,  $\theta^*$  is the bursting probability such that the capital price in the stationary equilibrium with stochastic bubbles is the same as that in the bubbleless equilibrium.

**Proposition 8** *Let condition (34) hold. If  $0 < \theta < \theta^*$ , then there exists a stationary equilibrium  $(B_s, Q_s, K_s)$  with stochastic bubbles such that  $K_s > K^*$ . In addition, if  $\theta$  is sufficiently small, then consumption also falls eventually after the bubble bursts.*

As in Weil (1987), a stationary equilibrium with stochastic bubbles exists if the probability that the bubble will burst is sufficiently small. In Weil's (1987) overlapping-generation model, the capital stock and output eventually rise after the bubble collapses. In contrast to his result, in our model the economy enters a recession after the bubble bursts in that consumption, capital and output fall eventually. The intuition is that the collapse of the bubble tightens the collateral constraint and impairs investment efficiency.

Proposition 8 compares the economy before the bubble collapses with the economy after the bubble collapses only in the steady state. It would be interesting to see what happens in the transition path. Since analytical results are not available, we solve the transition path

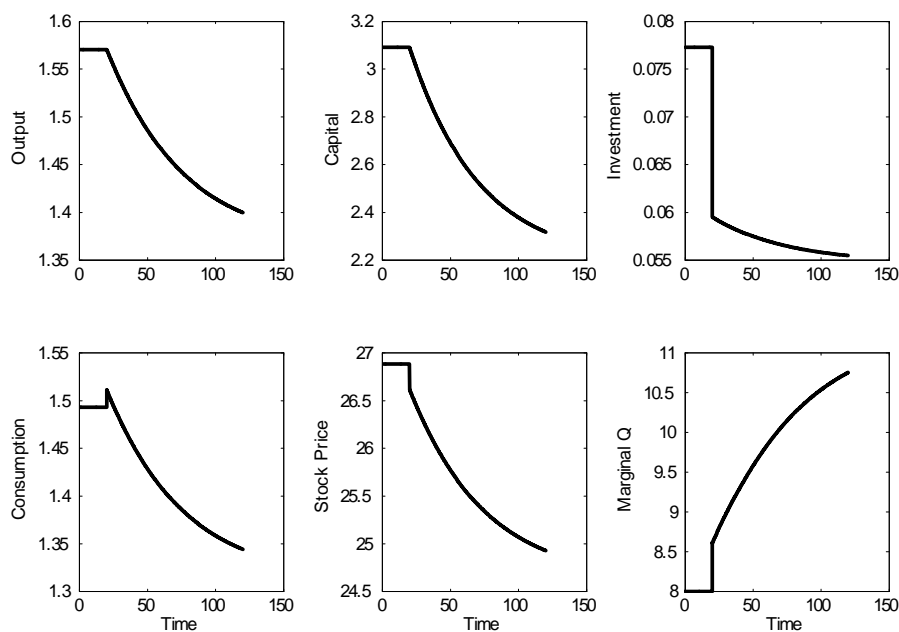


Figure 2: This figure plots the dynamics of the stationary equilibrium with stochastic bubbles. Assume that the bubble bursts at time  $t = 20$ . Set the parameter values as follows:  $r = 0.02$ ,  $\alpha = 0.4$ ,  $\delta = 0.025$ ,  $\theta = 0.05$ ,  $\pi = 0.01$ , and  $\xi = 0.2$ .

numerically and present the results in Figure 2.<sup>9</sup> In this numerical example, we assume that the bubble collapses at time  $t = 20$ . Immediately after the bubble collapses, investment falls discontinuously and then gradually decreases to its bubbleless steady-state level. But output and capital decrease continuously to their bubbleless steady-state levels. Consumption rises initially because of the fall of investment. But it quickly falls and then decreases to its bubbleless steady-state level. Importantly, the stock market crashes immediately after the bubble collapses in that the stock price drops discontinuously. Note that Tobin's marginal Q rises immediately after the bubble collapses and then gradually rises to its bubbleless steady-state value. This reflects the fact that the capital stock gradually falls after the bubble collapses. Note that marginal Q is not equal to average Q given the constant-returns-to-scale assumption in our model because of the presence of bubbles.

<sup>9</sup>We set some arbitrary parameter values as in Figure 2. The model is stylized and cannot be calibrated to match the data.

## 7 Public Assets and Credit Policy

We have shown that the collapse of bubbles generates a recession. Is there a government policy that restores economic efficiency? The inefficiency of our model comes from the credit constraints. In our model, firms can use internal funds and external loans to finance investment. External loans are subject to collateral constraints. Bubbles help relax these constraints, while the collapse of bubbles tightens them.

Now we suppose that the government can supply liquidity to the firms by issuing public bonds. These bonds are backed by lump-sum taxes. Households and firms can buy and sell these bonds. Firms can also use them as collateral to relax their collateral constraints. Let the government bonds supplied to the firms be  $M_t$  and the bond price be  $P_t$ . We start with the discrete-time environment described as in Section 2. The value of the government assets satisfies:

$$M_t P_t = T_t dt + M_{t+dt} P_t, \quad (50)$$

where  $T_t$  denotes lump-sum taxes. Taking the continuous-time limits yields:

$$\dot{M}_t P_t = -T_t. \quad (51)$$

It is more convenient to define  $D_t = P_t M_t$ . Then use the fact that  $\dot{D}_t = \dot{P}_t M_t + \dot{M}_t P_t$  to rewrite (51) as:

$$\dot{D}_t - \dot{P}_t M_t = \dot{D}_t - \frac{\dot{P}_t}{P_t} D_t = -T_t \text{ if } P_t > 0. \quad (52)$$

Since households are assumed to be risk neutral, from their optimization problem, we immediately obtain the asset pricing equation for the government bonds:  $P_t = e^{-rdt} P_{t+dt}$ . Taking the continuous-time limit yields:

$$\dot{P}_t = r P_t, \quad (53)$$

which implies that the growth rate of the government bond price is equal to the interest rate. Now we turn to firms' optimization problem below.

### 7.1 Equilibrium after the Bubble Bursts

We solve firms' dynamic optimization problem by dynamic programming. We start with the case in which the bubble has collapsed. Because firms can trade public assets, holdings of

government bonds are another state variable. We write firm  $j$ 's dynamic programming problem as follows:

$$\begin{aligned} V_t^*(K_t^j, M_t^j) = & \max R_t K_t^j dt - \pi I_t^j dt + P_t(M_t^j - M_{t+dt}^j) \\ & + e^{-rdt} V_{t+dt}^*((1 - \delta dt) K_t^j + I_t^j, M_{t+dt}^j) \pi dt \\ & + e^{-rdt} V_{t+dt}^*((1 - \delta dt) K_t^j, M_{t+dt}^j) (1 - \pi dt), \end{aligned} \quad (54)$$

subject to (5) and

$$L_t^j \leq e^{-rdt} V_{t+dt}^*(\xi K_t^j, 0) + P_t M_t^j, \quad (55)$$

$$M_{t+dt}^j \geq 0, \quad (56)$$

where  $M_t^j$  denotes the amount of government assets held by firm  $j$ . In equilibrium,  $\int M_t^j dj = M_t$ . Equation (55) indicates that firms use government assets as collateral. The expression  $e^{-rdt} V_{t+dt}^*(\xi K_t^j, 0)$  gives the market value of the collateralized assets  $\xi K_t^j$ . Equation (56) is a short-sale constraint, which rules out Ponzi schemes for public assets.

As in Section 3, we conjecture the value function takes the form:

$$V_t^*(K_t^j, M_t^j) = v_t^* K_t^j + v_t^{*M} M_t^j, \quad (57)$$

where  $v_t^*$  and  $v_t^{*M}$  are to be determined variables, that are independent of  $K_t^j$  or  $M_t^j$ . Because bubbles have collapsed, there is no bubble term in this conjecture. We define

$$Q_t^* = e^{-rdt} v_{t+dt}^*, \quad Q_t^{*M} = e^{-rdt} v_{t+dt}^{*M}.$$

We can then rewrite (54) as:

$$\begin{aligned} v_t^* K_t^j + v_t^{*M} M_t^j = & \max R_t K_t^j dt - \pi I_t^j dt + P_t(M_t^j - M_{t+dt}^j) \\ & + Q_t^*(1 - \delta dt) K_t^j + \pi Q_t^* I_t^j dt + Q_t^{*M} M_{t+dt}^j, \end{aligned} \quad (58)$$

subject to

$$0 \leq I_t^j \leq R_t K_t^j + \xi Q_t^* K_t^j + P_t M_t^j, \quad (59)$$

$$M_{t+dt}^j \geq 0. \quad (60)$$

When  $Q_t^* > 1$ , optimal investment achieves the upper bond. For an interior solution for the optimal holdings of government assets to exist, we must have  $P_t = Q_t^{*M}$ . Matching coefficients of  $K_t^j$  and  $M_t^j$  as well as the constant terms on the two sides of (58), we obtain (13), (14), and

$$v_t^{*M} = P_t + (Q_t^* - 1)P_t \pi dt.$$

As in Proposition 2, we may conduct aggregation and derive the equilibrium system for  $(P_t, Q_t^*, K_t)$  in the discrete time case. As in Proposition 3, the continuous-time limits satisfy the following differential equations:

$$\dot{P}_t = rP_t - P_t\pi(Q_t^* - 1), \quad (61)$$

$$\dot{K}_t = -\delta K_t + \pi(R_t K_t + \xi Q_t^* K_t + P_t M_t), \quad K_0 \text{ given}, \quad (62)$$

and an equation analogous to (19) for  $Q_t^*$ . In addition, the transversality condition

$$\lim_{T \rightarrow \infty} e^{-rT} P_T M_T = 0,$$

and other transversality conditions for  $K_t$  and  $B_t$  as in Proposition 3 must be satisfied. Here, we omit the detailed derivation of these conditions and the above differential equations.

Equation (61) is an asset pricing equation from the firms' trading. It is identical to the asset pricing equation (18) for the bubble. This is because the government bonds and the bubble on firm assets play the same role for the firms in that both of them can be used to relax the collateral constraints. The dividend yield of the government bonds to the firms is equal to  $\pi(Q_t^* - 1)$  when the bond price is positive. By contrast, there is no dividend yield to the households, as revealed by equation (53).

Comparing (53) with (61), we deduce that  $Q_t^* = 1$ . Substituting it into equation (19) reveals that  $R_t = r + \delta$ . This equation gives the efficient capital stock  $K_E$  for all time  $t$ . To support this capital stock in equilibrium, we need the value of the government debt  $D_t = P_t M_t$  to satisfy equation (62) for  $K_t = K_E$ . Solving yields:

$$D_t = D \equiv K_E \left( \delta \frac{1 - \pi}{\pi} - r - \xi \right) > 0 \quad (63)$$

By equations (53) and (52), we deduce that the lump-sum taxes must satisfy  $T_t = T \equiv rD$  for all  $t$ .

## 7.2 Equilibrium before the Bubble Bursts

Now, we turn to the equilibrium before the bubble bursts. We have to modify the dynamic programming problem (35) by incorporating trading of government bonds. By an analysis similar to that in the previous subsection and in Section 6, we can derive the continuous-time equilibrium system for  $(P_t, B_t, Q_t, K_t)$  before the bubble collapse. This system is given by equations (61), (45), (46) and

$$\dot{K}_t = -\delta K_t + \pi(R_t K_t + \xi Q_t K_t + B_t + P_t M_t), \quad K_0 \text{ given}. \quad (64)$$

By a no-arbitrage argument similar to that in the previous subsection, we deduce that  $Q_t = 1$ . By equation (46) and  $Q_t^* = 1$ , we deduce that  $R_t = r + \delta$ , which gives the efficient capital stock  $K_E$ . In addition,  $Q_t = 1$  and equation (45) imply that  $B_t = 0$  for all  $t$ . The bubble on the firm assets cannot be sustained in equilibrium because its dividend yield is zero and thus its growth rate is equal to  $r + \delta$ , which is higher than the zero rate of economic growth. Equation (64) gives the value of the government debt  $D_t = P_t M_t$  that supports the above efficient allocation.

We summarize the above analysis in the following proposition and relegate its detailed proof in the appendix.

**Proposition 9** *Suppose assumption (34) holds. Let the government issue a constant value  $D$  of government debt given by (63), which is backed by lump-sum taxes  $T_t = T \equiv rD$  for all  $t$ . Then this credit policy will eliminate the bubble on firm assets and make the economy achieve the efficient allocation.*

This proposition indicates that the government can design a credit policy that eliminates bubbles and achieves the efficient allocation. The key intuition is that the government may provide sufficient liquidity to firms so that firms do not need to rely on bubbles to relax credit constraints. The government plays the role of financial intermediaries by transferring funds from households to firms directly so that firms can overcome credit constraints. The government bond is a store of value and can also generate dividends to firms. The dividend yield is equal to the net benefit from new investment. For households, the government bond is just a store of value. No arbitrage forces the dividend yield to zero, which implies that the capital price must be equal to one. As a result, the economy can achieve the efficient allocation.

To implement the above policy. The government constantly retires the public bonds at the interest rate in order to maintain the total bond value constant. To back the government bonds, the government levies constant lump-sum taxes equal to the interest payments of bonds.

### 7.3 Discussion

An important part of the above credit policy is that the public bonds must be backed by lump-sum taxes. What will happen if they are unbacked assets? In this case, equation (50) implies that  $M_t$  is constant over time since  $T_t = 0$ . We thus normalize  $M_t = 1$  for all  $t$ . Our previous asset pricing equations for public bonds still apply here. Thus, if  $P_t > 0$ , then  $Q_t = Q_t^* = 1$ ,

which implies that  $K_t = K_E$  for all  $t$ . However, the capital accumulation equations (62) and (64) imply that the public bond price  $P_t$  must be constant over time, contradicting with the asset pricing equations for bonds. Thus, in equilibrium  $P_t = 0$ . The intuition is that the public bond is a bubble when it is an unbacked asset. Its rate of return or its growth rate is equal to the interest rate which is higher than the zero economic growth rate. Thus, the bubble cannot sustain in equilibrium.

## 8 Conclusion

In this paper, we provide an infinite-horizon model of a production economy with bubbles, in which firms meet stochastic investment opportunities and face credit constraints. Capital is not only an input for production, but also serves as collateral. We show that bubbles on this reproducible asset may arise, which relax collateral constraints and improve investment efficiency. The collapse of bubbles leads to a recession, even though there is no exogenous shock to the fundamental of the economy. Immediately after the collapse, investment falls discontinuously and the stock market crashes in that the stock price falls discontinuously. In the long run, output, investment, consumption, and capital all fall to their bubbleless steady-state values. We show that there is a credit policy that can eliminate the bubble on firm assets and can achieve the efficient allocation.

We focus on firms' credit constraints, but not on households' borrowing constraints. In addition, we consider complete markets economies in which all firm assets are publicly traded in a stock market. We study bubbles on these assets. Thus, our analysis provides a theory of the creation and collapse of stock price bubbles. Our analysis differs from most studies in the existing literature that analyze bubbles on intrinsically useless assets or on assets with exogenously given rents or dividends. In future research, it would be interesting to consider households' borrowing constraints or incomplete markets economies and then study the role of bubbles in this kind of environments. Finally, there is no economic growth in the present paper. In our work in progress, Miao and Wang (2011) extends the present paper to study endogenous growth.

## A Appendix

**Proof of Proposition 1:** Substituting the conjecture (9) into (4) and (6) yields:

$$v_t K_t^j + b_t = \max R_t K_t^j dt - \pi I_t^j dt + \pi e^{-rdt} v_{t+dt} K_{t+dt}^j dt \quad (65)$$

$$+ (1 - \pi dt) e^{-rdt} v_{t+dt} (1 - \delta dt) K_t^j + B_t,$$

$$L_t^j \leq \xi e^{-rdt} v_{t+dt} K_t^j + B_t, \quad (66)$$

where  $B_t$  is defined in (11) and  $K_{t+dt}^j$  satisfies (3) for the case with the arrival of the investment opportunity. We combine (5) and (66) to obtain:

$$0 \leq I_t^j \leq R_t K_t^j + \xi e^{-rdt} v_{t+dt} K_t^j + B_t. \quad (67)$$

Let  $Q_t$  be the Lagrange multiplier associated with (3) for the case with the arrival of the investment opportunity. The first-order condition with respect to  $K_{t+dt}^j$  delivers equation (12). When  $Q_t > 1$ , we obtain the optimal investment rule in (10). Plugging (10) and (3) into the Bellman equation (65) and matching coefficients of  $K_t^j$  and the terms unrelated to  $K_t^j$ , we obtain (13) and (14). Q.E.D.

**Proof of Proposition 2:** Using the optimal investment rule in (10) and aggregating equation (3), we obtain the aggregate capital accumulation equation (17) and the aggregate investment equation (21). Substituting (14) into (11) yields (15). Substituting (13) into (12) yields (16). The first-order condition for the static labor choice problem (1) gives  $w_t = (1 - \alpha) (K_t^j / N_t^j)^\alpha$ . We then obtain (2) and  $K_t^j = N_t^j (w_t / (1 - \alpha))^{1/\alpha}$ . Thus, the capital-labor ratio is identical for each firm. Aggregating yields  $K_t = N_t (w_t / (1 - \alpha))^{1/\alpha}$ . Using this equation to substitute out  $w_t$  in (2) yields  $R_t = \alpha K_t^{\alpha-1} N_t^{1-\alpha} = \alpha K_t^{\alpha-1}$  since  $N_t = 1$  in equilibrium. Aggregate output satisfies

$$Y_t = \int (K_t^j)^\alpha (N_t^j)^{1-\alpha} dj = \int (K_t^j / N_t^j)^\alpha N_t^j dj = (K_t^j / N_t^j)^\alpha \int N_t^j dj = K_t^\alpha N_t^{1-\alpha}.$$

This completes the proof. Q.E.D.

**Proof of Proposition 3:** By equation (17),

$$\frac{K_{t+dt} - K_t}{dt} = -\delta K_t + [R_t + \xi Q_t K_t + B_t] \pi.$$

Taking limit as  $dt \rightarrow 0$  yields equation (20). Using the approximation  $e^{r dt} = 1 + r dt$  in equation (15) yields:

$$B_t(1 + r dt) = B_{t+dt} [1 + \pi(Q_{t+dt} - 1)dt].$$

Simplifying yields:

$$\frac{B_t - B_{t+dt}}{dt} + r B_t = B_{t+dt} \pi(Q_{t+dt} - 1).$$

Taking limits as  $dt \rightarrow 0$  yields equation (18). Finally, we approximate equation (16) by:

$$Q_t(1 + r dt) = R_{t+dt} dt + (1 - \delta dt) Q_{t+dt} + (R_{t+dt} + \xi Q_{t+dt}) (Q_{t+dt} - 1) \pi dt.$$

Simplifying yields:

$$\frac{Q_t - Q_{t+dt}}{dt} + r Q_t = R_{t+dt} - \delta Q_{t+dt} + (R_{t+dt} + \xi Q_{t+dt}) (Q_{t+dt} - 1) \pi.$$

Taking limit as  $dt \rightarrow 0$  yields equation (19).

We may start with a continuous-time formulation directly. The Bellman equation in continuous time satisfies:

$$\begin{aligned} rV(K^j, S) &= \max_{I^j} RK^j - \pi I^j + \pi [V(K^j + I^j, S) - V(K^j, S)] \\ &\quad - \delta K^j V_K(K^j, S) + V_S(K^j, S) \dot{S}, \end{aligned}$$

where  $S = (B, Q)$  represents the vector of aggregate state variables. We may derive this Bellman equation by taking limits in (4) as  $dt \rightarrow 0$ . Conjecture  $V(K^j, B, Q) = QK^j + B$ . We can then solve the above Bellman equation. After aggregation, we can derive the system of differential equations in the proposition. Q.E.D.

**Proof of Proposition 4:** (i) The social planner solves the following problem:

$$\max_{I_t} \int_0^\infty e^{-rt} (K_t^\alpha - \pi I_t) dt$$

subject to

$$\dot{K}_t = -\delta K_t + \pi I_t, \quad K_0 \text{ given}$$

where  $K_t$  is the aggregate capital stock and  $I_t$  is the investment level for each firm with the arrival of the investment opportunity. From this problem, we can derive the efficient capital stock  $K_E$ , which satisfies  $\alpha (K_E)^{\alpha-1} = r + \delta$ . The efficient output, investment and consumption levels are given by  $Y_E = (K_E)^\alpha$ ,  $I_E = \delta/\pi K_E$ , and  $C_E = (K_E)^\alpha - \delta K_E$ , respectively.

From the proof of Proposition 1, we can rewrite (65) as:

$$v_t K_t^j = \max R_t K_t^j dt - \pi I_t^j dt + Q_t(1 - \delta dt)K_t^j + Q_t \pi I_t^j dt. \quad (68)$$

Suppose assumption (24) holds. We conjecture  $Q^* = 1$  and  $Q_t = 1$ . Substituting this conjecture into the above equation and matching coefficients of  $K_t^j$  give:

$$v_t = R_t dt + 1 - \delta dt.$$

Since  $Q_t = e^{-rdt} v_{t+dt} = 1$ , we have  $e^{rdt} = R_{t+dt} dt + 1 - \delta dt$ . Approximating this equation yields:

$$1 + rdt = R_{t+dt} dt + 1 - \delta dt.$$

Taking limits as  $dt \rightarrow 0$  gives  $R_t = r + \delta = \alpha K_t^{*\alpha-1}$ . Thus,  $K_t^* = K_E$ . Given this constant capital stock for all firms, the optimal investment level satisfies  $\delta K_t^* = \pi I_t^*$ . Thus,  $I_t^*/K_t^* = \delta/\pi$ . We can easily check that assumption (24) implies that

$$\frac{\delta}{\pi} = I_t^*/K_t^* \leq R_t + \xi = r + \delta + \xi.$$

Thus, the investment constraint (5) or (67) is satisfied for  $Q_t = 1$  and  $B_t = 0$ . We conclude that the solutions  $Q_t = 1$ ,  $K_t^* = K_E$ , and  $I_t^*/K_t^* = \delta/\pi$  give the bubbleless equilibrium, which also delivers the efficient allocation.

(ii) Suppose (25) holds. Conjecture  $Q_t > 1$  in some neighborhood of the bubbleless steady state. We can then apply Proposition 3 and derive the steady-state equations (22) and (23). From these equation, we obtain the steady-state solution  $Q^*$  and  $K^*$  in (26) and (27), respectively. Assumption (25) implies that  $Q^* > 1$ . By continuity,  $Q_t > 1$  in some neighborhood of  $(Q^*, K^*)$ . This verifies our conjecture. Q.E.D.

**Proof of Proposition 5:** Solving equations (22), (29), (30) yields equations (31)-(33). By (31),  $B > 0$  if and only if (34) holds. From (26) and (32), we deduce that  $Q_b < Q^*$ . Using condition (34), it is straightforward to check that  $K_{GR} > K_E > K_b > K^*$ . From (31), it is also straightforward to verify that the bubble-asset ratio  $B/K_b$  decreases with  $\xi$ . Q.E.D.

**Proof of Proposition 6:** First, we consider the log-linearized system around the bubbly steady state  $(B, Q_b, K_b)$ . We use  $\hat{X}_t$  to denote the percentage deviation from the steady state

value for any variable  $X_t$ , i.e.,  $\hat{X}_t = \ln X_t - \ln X$ . We can show that log-linearized system is given by:

$$\begin{bmatrix} d\hat{B}_t/dt \\ d\hat{Q}_t/dt \\ d\hat{K}_t/dt \end{bmatrix} = A \begin{bmatrix} \hat{B}_t \\ \hat{Q}_t \\ \hat{K}_t \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & -(r + \pi) & 0 \\ 0 & \delta - \frac{(r + \delta + \xi)(r + \pi)}{1 + r} & [(1 - \xi)r + \delta](1 - \alpha) \\ \pi B/K_b & \xi(r + \pi) & -(\pi R_b(1 - \alpha) + \pi B/K_b) \end{bmatrix}. \quad (69)$$

We denote this matrix by:

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & b & c \\ d & e & f \end{bmatrix},$$

where we deduce from (69) that  $a < 0$ ,  $b > 0$ ,  $c > 0$ ,  $d > 0$ ,  $e > 0$ , and  $f < 0$ . We compute the characteristic equation for the matrix  $A$ :

$$F(x) \equiv x^3 - (b + f)x^2 + (bf - ce)x - acd = 0. \quad (70)$$

We observe that  $F(0) = -acd > 0$  and  $F(-\infty) = -\infty$ . Thus, there exists a negative root to the above equation, denoted by  $\lambda_1 < 0$ . Let the other two roots be  $\lambda_2$  and  $\lambda_3$ . We rewrite  $F(x)$  as:

$$\begin{aligned} F(x) &= (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) \\ &= x^3 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)x - \lambda_1\lambda_2\lambda_3. \end{aligned} \quad (71)$$

Matching terms in equations (70) and (71) yields  $\lambda_1\lambda_2\lambda_3 = acd < 0$  and

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = bf - cd < 0. \quad (72)$$

We consider two cases. (i) If  $\lambda_2$  and  $\lambda_3$  are two real roots, then it follows from  $\lambda_1 < 0$  that  $\lambda_2$  and  $\lambda_3$  must have the same sign. Suppose  $\lambda_2 < 0$  and  $\lambda_3 < 0$ , we then have  $\lambda_1\lambda_2 > 0$  and  $\lambda_1\lambda_3 > 0$ . This implies that  $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0$ , which contradicts equation (72). Thus, we must have  $\lambda_2 > 0$  and  $\lambda_3 > 0$ .

(ii) If one of  $\lambda_2$  and  $\lambda_3$  is complex, then the other must be also complex. Let

$$\lambda_2 = g + hi \text{ and } \lambda_3 = g - hi,$$

where  $g$  and  $h$  are some real numbers. We can show that

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 2g\lambda_1 + g^2 + h^2.$$

Since  $\lambda_1 < 0$ , the above equation and equation (72) imply that  $g > 0$ .

From the above analysis, we conclude that the matrix  $A$  has one negative eigenvalues and the other two eigenvalues are either positive real numbers or complex numbers with positive real part. As a result, the bubbly steady state is a local saddle point and the stable manifold is one dimensional.

Next, we consider the local dynamics around the bubbleless steady state  $(0, Q^*, K^*)$ . We linearize  $B_t$  around zero and log-linearize  $Q_t$  and  $K_t$  and obtain linearized system:

$$\begin{bmatrix} dB_t/dt \\ d\hat{Q}_t/dt \\ d\hat{K}_t/dt \end{bmatrix} = J \begin{bmatrix} B_t \\ \hat{Q}_t \\ \hat{K}_t \end{bmatrix},$$

where

$$J = \begin{bmatrix} r - \pi(Q^* - 1) & 0 & 0 \\ 0 & a & b \\ \frac{\pi}{K^*} & c & d \end{bmatrix},$$

where

$$\begin{aligned} a &= \frac{R^*}{Q^*}[1 + \pi(Q^* - 1)] - \left(\frac{R^*}{Q^*} + \xi\right)\pi Q^*, \\ b &= \frac{R^*}{Q^*}[1 + \pi(Q^* - 1)](1 - \alpha) > 0, \\ c &= \pi\xi Q^* > 0, \\ d &= \pi R^*[\alpha - 1] < 0. \end{aligned}$$

Using a similar method for the bubbly steady state, we analyze the three eigenvalues of the matrix  $J$ . One eigenvalue, denoted by  $\lambda_1$ , is equal to  $r - \pi(Q^* - 1) < 0$  and the other two, denoted by  $\lambda_2$  and  $\lambda_3$ , satisfy

$$\lambda_2\lambda_3 = ad - bc. \quad (73)$$

Notice that we have

$$\frac{a}{b} = \frac{1}{1 - \alpha} \left[ \frac{1 - \pi}{1 + \pi(Q^* - 1)} - \xi \frac{Q^*}{R^*} \frac{\pi Q^*}{1 + \pi(Q^* - 1)} \right],$$

and

$$\frac{c}{d} = -\frac{\xi Q^*}{R^*} \frac{1}{1 - \alpha}.$$

So we have

$$\frac{a}{b} - \frac{c}{d} > 0 \text{ or } \frac{a}{b} > \frac{c}{d}.$$

Since  $b > 0$  and  $d < 0$ , we deduce that  $ad < cb$ . It follows from (73) that  $\lambda_2\lambda_3 < 0$ , implying that  $\lambda_2$  and  $\lambda_3$  must be two real numbers with opposite signs. We conclude that the bubbleless steady state is a local saddle point and the stable manifold is two dimensional. Q.E.D.

**Proof of Proposition 7:** As we discussed in the main text, we may derive equations (43) and (44). Substituting equation (43) into (41) and using the definition  $Q_t^* = e^{-rdt}v_{t+dt}^*$ , we can derive that:

$$\begin{aligned} Q_t &= \theta Q_t^* dt + e^{-rdt}(1 - \theta dt)[R_{t+dt}dt + Q_{t+dt}(1 - \delta dt) \\ &\quad + \pi(Q_{t+dt} - 1)(R_{t+dt} + Q_{t+dt}\xi)dt]. \end{aligned} \quad (74)$$

Using the approximation  $e^{-rdt} = 1 - rdt$  and removing all terms that have orders at least  $dt^2$ , we approximate the above equation by:

$$\begin{aligned} Q_t - Q_{t+dt} &= \theta Q_t^* dt + R_{t+dt}dt - \delta Q_{t+dt}dt + \pi(Q_{t+dt} - 1)(R_{t+dt} + \xi Q_{t+dt})dt \\ &\quad - (r + \theta)Q_{t+dt}dt. \end{aligned} \quad (75)$$

Dividing by  $dt$  on the two sides and taking limits as  $dt \rightarrow 0$ , we obtain:

$$-\dot{Q}_t = R_t - (r + \delta + \theta)Q_t + \theta Q_t^* + \pi(Q_t - 1)(R_t + \xi Q_t), \quad (76)$$

which gives equation (46). Similarly, substituting equation (44) into (42) and taking limits, we can derive equation (45). Q.E.D.

**Proof of Proposition 8:** Let  $Q(\theta)$  be the expression on the right-hand side of equation (47). We then use this equation to rewrite equation (48) as:

$$\alpha K^{\alpha-1}(1 + r + \theta) - (r + \delta + \theta)Q(\theta) + \theta g(K) + (r + \theta)\xi Q(\theta) = 0.$$

Define the function  $F(K; \theta)$  as the expression on the left-hand side of the above equation. Notice  $Q(\theta^*) = Q^* = g(K^*)$  by definition and  $Q(0) = Q_b$  where  $Q_b$  is given in (32). The condition (34) ensures the existence of the bubbly steady-state value  $Q_b$  and the bubbleless steady-state values  $Q^*$  and  $K^*$ .

Define

$$K_{\max} = \max_{0 \leq \theta \leq \theta^*} \left[ \frac{(r + \delta + \theta - (r + \theta)\xi)Q(\theta) - \theta Q^*}{\alpha(1 + r + \theta)} \right]^{\frac{1}{\alpha-1}}.$$

By (33), we can show that

$$K_b = \left[ \frac{(r + \delta - r\xi)Q(0)}{\alpha(1+r)} \right]^{\frac{1}{\alpha-1}}.$$

Thus, we have  $K_{\max} \geq K_b$  and hence  $K_{\max} > K^*$ . We want to prove that

$$F(K^*; \theta) > 0, \quad F(K_{\max}; \theta) < 0,$$

for  $\theta \in (0, \theta^*)$ . If this true, then it follows from the intermediate value theorem that there exists a solution  $K_s$  to  $F(K; \theta) = 0$  such that  $K_s \in (K^*, K_{\max})$ .

First, notice that

$$\begin{aligned} F(K^*, 0) &= \alpha K^{*\alpha-1}(1+r) - r(1-\xi)Q_b - \delta Q_b \\ &> \alpha K_b^{\alpha-1}(1+r) - r(1-\xi)Q_b - \delta Q_b \\ &= 0, \end{aligned}$$

and

$$F(K^*, \theta^*) = 0.$$

We can verify that  $F(K; \theta)$  is concave in  $\theta$  for any fixed  $K$ . Thus, for all  $0 < \theta < \theta^*$ ,

$$\begin{aligned} F(K^*; \theta) &= F\left(K^*, \left(1 - \frac{\theta}{\theta^*}\right)0 + \frac{\theta}{\theta^*}\theta^*\right) \\ &> \left(1 - \frac{\theta}{\theta^*}\right)F(K^*, 0) + \frac{\theta}{\theta^*}F(K^*, \theta^*) \\ &> 0. \end{aligned}$$

Next, for  $K \in (K^*, K_{\max})$ , we derive the following:

$$\begin{aligned} F(K_{\max}; \theta) &= \alpha K_{\max}^{\alpha-1}(1+r+\theta) - (r+\delta+\theta)Q(\theta) + \theta g(K_{\max}) + (r+\theta)\xi Q(\theta) \\ &< \alpha K_{\max}^{\alpha-1}(1+r+\theta) - (r+\delta+\theta)Q(\theta) + \theta g(K^*) + (r+\theta)\xi Q(\theta) \\ &< 0, \end{aligned}$$

where the first inequality follows from the fact that the saddle path for the bubbleless equilibrium is downward sloping as illustrated in Figure 1 so that  $g(K_{\max}) < g(K^*)$ , and the second inequality follows from the definition of  $K_{\max}$  and the fact that  $g(K^*) = Q^*$ .

Finally, note that  $Q(\theta) < Q^*$  for  $0 < \theta < \theta^*$ . We use equation (30) and  $K_s > K^*$  to deduce that

$$\begin{aligned}\frac{B_s}{K_s} &= \frac{\delta}{\pi} - \alpha K_s^{\alpha-1} - \xi Q(\theta) \\ &> \frac{\delta}{\pi} - \alpha K^{*\alpha-1} - \xi Q^* \\ &= 0.\end{aligned}$$

This completes the proof of the existence of stationary equilibrium with stochastic bubbles  $(B_s, Q_s, K_s)$ .

When  $\theta = 0$ , the bubble never bursts and hence  $K_s = K_b$ . When  $\theta$  is sufficiently small,  $K_s$  is close to  $K_b$  by continuity. Since  $K_b$  is less than the golden rule capital stock  $K_{GR}$ ,  $K_s < K_{GR}$  when  $\theta$  is sufficiently small. Since  $K^\alpha - \delta K$  is increasing for all  $K < K_{GR}$ , we deduce that  $K_s^\alpha - \delta K_s > K^{*\alpha} - \delta K^*$ . This implies that the consumption level before the bubble collapses is higher than the consumption level in the steady state after the bubble collapses. Q.E.D.

**Proof of Proposition 9:** We write the firm's dynamic programming before the bubble collapses as:

$$\begin{aligned}V_t(K_t^j, M_t^j) &= \max R_t K_t^j dt - \pi I_t^j dt + P_t M_t^j - P_t M_{t+dt}^j \\ &\quad + e^{-rdt} (1 - \theta dt) V_{t+dt}((1 - \delta dt) K_t^j + I_t^j, M_{t+dt}^j) \pi dt \\ &\quad + e^{-rdt} (1 - \theta dt) V_{t+dt}((1 - \delta dt) K_t^j, M_{t+dt}^j) (1 - \pi dt) \\ &\quad + e^{-rdt} \theta dt V_{t+dt}^*((1 - \delta dt) K_t^j + I_t^j, M_{t+dt}^j) \pi dt \\ &\quad + e^{-rdt} \theta dt V_{t+dt}^*((1 - \delta dt) K_t^j, M_{t+dt}^j) (1 - \pi dt),\end{aligned}\tag{77}$$

subject to (5),  $M_{t+dt}^j \geq 0$ , and

$$L_t^j \leq e^{-rdt} V_{t+dt}(\xi K_t^j, 0) (1 - \theta dt) + e^{-rdt} V_{t+dt}^*(\xi K_t^j, 0) \theta dt + P_t M_t.\tag{78}$$

We conjecture that the value function takes the form:

$$V_t(K_t^j, M_t^j) = v_t K_t^j + v_t^M M_t^j + b_t,$$

where  $v_t, v_t^M$ , and  $b_t$  are to be determined variables independent of  $j$ . Define  $Q_t$  and  $B_t$  as in (41) and (42), respectively, and define

$$Q_t^M = e^{-rdt} [(1 - \theta dt) v_{t+dt}^M + v_{t+dt}^{*M} \theta dt].$$

By an analysis similar to that in Section 7.1, we can derive the continuous-time limiting system for  $(P_t, B_t, Q_t, K_t)$  given in Section 7.2. Finally, we follow the procedure described there to establish Proposition 9. Q.E.D.

## References

- Benhabib, Jess and Farmer, Roger, 1999. Indeterminacy and sunspots in macroeconomics, in: Taylor, J., Woodford, M. (Eds.), *Handbook of Macroeconomics*, Vol. 1A, North-Holland, Amsterdam, 387–448.
- Bernanke, Ben, and Mark Gertler, 1989, Agency Costs, Net Worth, and Business Fluctuations, *American Economic Review* 79(1), 14-31.
- Bernanke, Ben, Mark Gertler and Simon Gilchrist, 1999, Financial Accelerator in a Quantitative Business Cycle Framework, In: J. Taylor and M. Woodford (eds.), *Handbook of Macroeconomics*, Elsevier Science, Amsterdam.
- Blanchard, Olivier, and Mark Watson, 1982, Bubbles, Rational Expectations and Financial Markets, Harvard Institute of Economic Research Working Paper No. 945.
- Brunnermeier, Markus, 2009, Bubbles, entry in *The New Palgrave Dictionary of Economics*, edited by Steven Durlauf and Lawrence Blume, 2nd edition.
- Caballero, Ricardo J. and Arvind Krishnamurthy, 2006, Bubbles and Capital Flow Volatility: Causes and Risk Management, *Journal of Monetary Economics* 53, 35-53.
- Carlstrom, Charles T. and Timothy S. Fuerst, 1997, Agency Costs, Net Worth, and Business Fluctuations: A Computable General Equilibrium Analysis, *American Economic Review* 87, 893-910.
- Diamond, Peter, 1965, National Debt in a Neoclassical Growth Model, *American Economic Review* 55, 1126-1150.
- Farhi, Emmanuel and Jean Tirole, 2010, Bubbly Liquidity, working paper, Harvard University.
- Hayashi, Fumio, 1982, Tobin's Marginal  $q$  and Average  $q$ : A Neoclassical Interpretation, *Econometrica* 50, 213-224.
- Hellwig, Christian, and Guido Lorenzoni, 2009, Bubbles and Self-enforcing Debt, *Econometrica* 77, 1137-1164.
- Jermann, Urban, and Vincenzo Quadrini, 2010, Macroeconomic Effects of Financial Shocks, forthcoming in *American Economic Review*.
- Kiyotaki, Nobuhiro, and John Moore, 1997, Credit Cycles, *Journal of Political Economy* 105, 211-248.
- Kiyotaki, Nobuhiro, and John Moore, 2005, Liquidity and Asset Prices, *International Economic Review* 46, 317-349.
- Kiyotaki, Nobuhiro, and John Moore, 2008, Liquidity, Business Cycles and Monetary Policy, working paper, Princeton University.
- Kocherlakota, Narayana, 1992, Bubbles and Constraints on Debt Accumulation, *Journal of Economic Theory* 57, 245-256.

- Kocherlakota, Narayana, 1998, Injecting Rational Bubbles, *Journal of Economic Theory* 142, 218-232.
- Kocherlakota, Narayana, 2009, Bursting Bubbles: Consequences and Cures, mimeo, University of Minnesota.
- Martin, Alberto and Jaume Ventura, 2010, Theoretical Notes on Bubbles and the Current Crisis, working paper, Universitat Pompeu Fabra.
- Miao, Jianjun and Pengfei, Wang, 2010, Credit Risk and Business Cycles, working paper, Boston University.
- Miao, Jianjun and Pengfei, Wang, 2011, Bubbles, Credit Constraints, and Endogenous Growth, work in progress, Boston University.
- Santos, Manuel S. and Michael Woodford, 1997, Rational Asset Pricing Bubbles, *Econometrica* 65, 19-58.
- Scheinkman, Jose A. and Laurence Weiss, 1986, Borrowing Constraints and Aggregate Economic Activity, *Econometrica* 54, 23-46.
- Scheinkman, Jose, A. and Wei Xiong, 2003, Overconfidence and Speculative Bubbles, *Journal of Political Economy* 111, 1183-1219.
- Tirole, Jean, 1982, On the Possibility of Speculation under Rational Expectations, *Econometrica* 50, 1163-1182.
- Tirole, Jean, 1985, Asset Bubbles and Overlapping Generations, *Econometrica* 53, 1499-1528.
- Tobin, James, 1969, A General Equilibrium Approach to Monetary Policy, *Journal of Money, Credit, and Banking* 1, 15-29.
- Wang, Pengfei and Yi Wen, 2009, Speculative Bubbles and Financial Crisis, working paper, Federal Reserve Bank of St. Louis.
- Weil, Philippe, 1987, Confidence and the Real Value of Money in an Overlapping Generations Economy, *Quarterly Journal of Economics* 102, 1-22.