

# Informational Black Holes in Auctions

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## ABSTRACT

A central role for financial markets is to assess whether new projects are worth pursuing or not. We extend standard auction theory to capture this role by studying new venture financing. Paradoxically, when the information generated in the auction is valuable for making real investment decisions, the informational efficiency of the market is destroyed. To add to the paradox, as the number of market participants with useful information increases a growing share of them fall into an “informational black hole,” making markets even less efficient. Contrary to the predictions of standard auction theory, social surplus and seller revenues can be decreasing in the number of bidders, the linkage principle of [Milgrom and Weber \(1982\)](#) may not hold, collusion among investors may be beneficial for the seller, and the revenue-ranking of standard auction formats can be reversed.

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A central role for financial markets is to allocate resources to their most productive use. Can we expect markets to fulfill this role efficiently when information is dispersed among market participants? In this paper, we revisit this classic question in a canonical capital-raising setting where prospective entrepreneurs seek financing for new projects from investors. Investors have some expertise in evaluating projects (we can think of them as sophisticated actors such as venture capitalist, business angels, or commercial banks) and compete with each other for the right to finance the most promising projects. To create maximal surplus, markets should be informationally efficient so that only worthwhile projects get financed. Can we hope to accomplish this goal?

Based on the insights in auction theory, one might be tempted to answer “Yes”. It is well-known that in the standard setting, the ascending-price auction aggregates all the information possessed by bidders (Kremer (2002) and Han and Shum (2004)). The two other common auction formats, first-price and second-price auctions, have similar properties: although the equilibrium price does not aggregate information fully, the history of bids does, so that anyone who observes bids ex post should be able to make the right decision about whether to start a project or not. It appears that markets can solve the resource-allocation problem.

The message in our paper is a more pessimistic one. We show that none of the standard auction formats aggregate information properly. Strikingly, even when the market grows so large that as an aggregate it possesses perfect information about which projects are worth financing and which are not, there will be substantial allocational inefficiencies. This result is driven by our only modelling departure from the standard auction setting: In the standard setting, the existence of the asset being sold is taken as given. In our setting, there is uncertainty about whether the project is worthwhile, and the decision to pursue the project is made based on the information generated in the auction. This is a necessary ingredient in order to study whether the market does a good job in allocating resources. Paradoxically, the introduction of a real surplus-creating role for information can destroy the informational efficiency of the market.

This result is due to the effect of the winner’s curse on bidder participation in a setting where an asset often has zero value (which is the case for a project that does not get financed). With many potential bidders, the winner’s curse implies that a bidder with only moderately optimistic information about a project will assume that the project has negative net present value when he wins it, since winning implies that all other bidders are less optimistic. Hence, such bidders might as well not participate in the auction—they fall into a “black hole” where their information is lost. This informational black hole grows with the size of the market, because the winner’s curse gets stronger with the number of potential investors. Therefore, even if a very large

number of bidders are invited to an auction, only the very small fraction of them who have the most optimistic beliefs will actually submit a non-zero bid. Thus, the decision of whether to start the project or not will be relatively uninformed.

This insight has normative implications for how entrepreneurs should maximize revenues that drastically contrast with the prescriptions of standard auction theory. Common wisdom, based on the results of [Bulow and Klemperer \(1996\)](#) and [Bali and Jackson \(2002\)](#) among others, is that having more bidders is better, since increasing the number of bidders increases competition and reduces the informational rent going to bidders. Furthermore, one might expect the larger collective information set possessed by a larger set of investors to lead to better financing decisions and hence a higher surplus. In our setting, however, restricting the number of potential investors that are invited to submit a bid can improve the revenues of the entrepreneur. If fewer investors are invited, the winner's curse will be less severe, which implies that investors with somewhat less optimistic beliefs will also submit bids. Thus, the information generated in the auction will be drawn from a more representative sample. For common distributional assumptions, this improves the informational properties of the auction, which in turn improves surplus. Although the auction becomes less competitive, so that the entrepreneur might capture a somewhat smaller fraction of the surplus, we show that the positive effect on surplus is often large enough to also improve revenues of the entrepreneur. This result may explain why sellers often explicitly restrict the number of potential investors they approach for financing.

We also show that the famous “linkage principle” of [Milgrom and Weber \(1982\)](#) may fail in our setting. The linkage principle holds that any value-relevant information that can be revealed before an auction should be revealed in order to lower the informational rent of bidders. For example, if an entrepreneur can postpone seeking financing until some public information about market conditions is revealed, he should do so. In our setting, to the contrary, it is often better to attempt financing of the project *before* some value-relevant information is revealed. The reason is that residual uncertainty creates an option value to the project which makes less optimistic bidders participate, which in turn increases the information aggregation properties of the auction.

We discuss a number of strategies for improving the efficiency of the auction. We first study the effect of revealing bids after the auction but before the investment decision is made. Our first result is negative—none of the inefficient equilibria are affected by the revelation of bids. For the ascending-price auction, there are also no extra equilibria that can be supported when bids are revealed, since all bids can anyway be inferred by observing when bidders drop out of the auction. However, we show that revealing bids introduces new, more efficient equilibria in the first-price and second-

price auction. As the market grows large, some of these equilibria lead to first-best efficiency, full information aggregation, and higher revenues than in any equilibrium of the ascending-price auction. Thus, the result from standard auction theory that ascending-price auctions lead to the highest revenues can be violated in our setting.

Another way to improve efficiency is to allow a sufficiently large number of investors to receive a stake in the project, if this is practically feasible. As shown in [Pesendorfer and Swinkels \(1997\)](#), in a multi-unit auction where the number of units grows with the number of bidders, a loser’s curse balances out the winner’s curse which in our setting leads to higher participation and a recovery of information aggregation, and hence a higher surplus. This result may explain, for example, why IPO allocations are rationed to increase the number of winning participants.

A related solution is to allow syndicates or consortia consisting of multiple investors to submit joint “club bids” in the auction. Club bids are common practice in private equity settings, and have been the subject of investigation by competition authorities for creating anti-competitive collusion. Indeed, in a standard auction setting, club bids reduce the expected revenues of the seller. In our setting, the opposite may hold—because club bids reduce the winner’s curse problem, it encourages participation, which increases the efficiency of the auction.

We are not the first to study auction-like settings of project financing. [Broecker \(1990\)](#) derives a credit market equilibrium which is a special case of our model when first-price auctions are used, signals are binary, and banks who provide financing do not have the option to cancel a project after an offer is accepted. [Broecker \(1990\)](#) does not study information aggregation and surplus specifically and does not consider the effect of reducing the number of bidders, releasing information, revealing bids, or allowing bidders to endogenously decide on the investment after the auction is over.

Another paper closely related to ours is [Atakan and Ekmekci \(2014\)](#), who also show that information can fail in large markets when an action has to be taken after the auction is run. Our papers are complementary in that they provide quite different explanations for why information aggregation may fail, and in that they apply to quite different market settings. We focus on a project financing setting where only a limited number of bidders can end up as investors in a firm, and where only one decision can be taken for a given firm. [Atakan and Ekmekci \(2014\)](#) focus on a situation where a large set of existing assets are sold (such as different plots of land) and where the buyers of the assets can take different actions after the auction (owners of different plots of land can choose to grow different crops). The asset is always valuable but the optimal action depends on the aggregate state. This can lead to pooling (and hence information destruction) in equilibrium—bidders with relatively low signals will want to bid high

enough to get an allocation when the aggregate state is low, but low enough to avoid an allocation when the state is high, which pushes them towards the same pooling bid. The pooling equilibrium fails to exist if there are too few units for low bidders to have a reasonable chance of winning, or if all winners have to agree on an action, or if bids are revealed ex post, all of which are natural ingredients in the economic setting we study. [Atakan and Ekmekci \(2014\)](#) also do not study the effect of restricting the number of bidders, optimal timing of the release of private information, or the effect of collusion among bidders.

A few other papers also study auction settings where some decision has to be made about how to run the project up for sale. [Cong \(2014\)](#) and [Board \(2007\)](#) study private-value models of auctioning options, and focus on the efficiency of exercise decisions by winning bidders. Because information aggregation is unimportant in pure private value settings, their models are silent on the informational properties of auctions that are central to our analysis.

Our result that restricting the number of bidders can reduce revenues can also be found in [Bulow and Klemperer \(2002\)](#), but for a different reason. They study a situation in which bidder valuations depend on a common value component that is the sum of the independently drawn bidder signals, and a (very small) private value component. In this “sum of signals” model the expected auction revenues decreases with the difference in signals between the highest and second highest bidder, a difference that is smaller with fewer bidders for some distributions (such as the normal distribution).<sup>1</sup> In the more standard pure common value model that we study, where bidder signals are independent conditional on the state, revenues converge to the expected value of the asset with the number of bidders, and so more bidders are better if the asset value is exogenous. Despite this, revenues can go down with the number of bidders in our setting because the information loss due to the winner’s curse reduces the value of the asset. In contrast, in the setting of [Bulow and Klemperer \(2002\)](#) more bidders lead to better information aggregation and (slightly) higher asset values.

At a general level, our results are reminiscent of the [Grossman and Stiglitz \(1980\)](#) result on the impossibility of informationally efficient markets. They show that in a rational expectations equilibrium with costly information acquisition, prices cannot be fully revealing. Our result is stronger—we show in a fully specified game-theoretic setting that markets are informationally inefficient even when information is costless.

There is also a parallel to the literature on feedback effects between the price observed in secondary markets and real decisions, summarized in [Bond, Edmans and](#)

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<sup>1</sup>When bidders are asymmetric so that some bidders have a much larger private value than others and this is known ex ante, this result flips so that it is optimal to reduce the number of bidders exactly when it is not in the symmetric case.

Goldstein (2012). Bond and Goldstein (2014) show that when an economic actor takes real decisions based on the information in stock prices, they affect the incentives to trade on this information in an endogenous way that may destroy the allocational efficiency of the market. Edmans, Goldstein, Jiang (2014) show that negative news will be less likely to be incorporated in stock prices because firms may act on this information by cancelling negative NPV projects, rendering short positions less valuable. Our paper shows that informational and allocational efficiency can fail even in the primary market for capital, where investors directly bear the consequences of their actions.

## 1. Model setup

We consider a penniless entrepreneur seeking outside financing for a new project from a set of  $N$  potential investors indexed by  $i \in \{1, \dots, N\}$ .<sup>2</sup> All agents are risk neutral. The project requires one unit of investment, and can be of two types: good ( $G$ ) and bad ( $B$ ), where the unconditional probability of the project being good is  $\pi$ . If the project is good it pays  $1 + X$ . Otherwise, it pays 0.<sup>3</sup> We denote the net present value, or NPV, of the project by  $V$ , a random variable that takes value  $X$  if the project is good and value  $-1$  if the project is bad.

Investors observe private signals  $S_i \in [0, 1]$  drawn independently from a distribution with cumulative distribution function  $F_G(s)$  and density  $f_G(s)$  if the type is good, and from a distribution with cdf  $F_B(s)$  and density  $f_B(s)$  if the type is bad. We make the following assumption about the signal distribution:

ASSUMPTION 1: *Signals satisfy the monotone likelihood ratio property (MLRP):*

$$\forall s > s', \quad \frac{f_G(s)}{f_B(s)} \geq \frac{f_G(s')}{f_B(s')}.$$

Both  $f_G(s)$  and  $f_B(s)$  are continuously differentiable at  $s = 1$ ,  $f_B(1) > 0$ , and  $\lambda \equiv f_G(1)/f_B(1) > 1$ .

Without loss of generality, we will also assume that  $f_G(s)$  and  $f_B(s)$  are left-continuous and have right limits everywhere. Assumption 1 ensures that higher signals are at least weakly better news than lower signals. Assuming that densities are continuously differentiable at the top of the signal distribution simplifies our proof, but is not essential for our results.

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<sup>2</sup>Although we assume the entrepreneur has zero wealth to invest in the project, this is not essential for our results. Our results generalize to situations where the entrepreneur has either wealth or other assets to pledge against the project.

<sup>3</sup>All of our results generalize to the case where there is a continuous type space, and where cash flows can be random conditional on the type.

We denote the likelihood ratio at the top of the distribution by  $\lambda$ , a quantity that will be important in our asymptotic analysis. Assuming  $\lambda > 1$  ensures that MLRP is strict over a set of non-zero measure, which in turn implies that as  $N \rightarrow \infty$ , an observer of all signals would learn the true type with probability one. Therefore, for large enough  $N$ , the aggregate market information is valuable for making the right investment decision.

To focus our analysis on the most interesting case, we make the stronger assumption that the signal of a single investor can be sufficiently pessimistic for the expected value of the project to be negative:

ASSUMPTION 2:  $E(V|S_i = 0) < 0$ .

Assumption 2 is not essential for our results — what matters is that the project becomes negative NPV conditional on a joint observation of the lowest signal by some critical lower bound on the number of investors, which is already guaranteed by Assumption 1. When there are fewer investors, the project would always be worth pursuing, so our setting would be equivalent to the standard setting of [Milgrom and Weber \(1982\)](#) where an exogenous asset is sold, and information aggregation, surplus and revenues would all be improved by increasing the number of bidders. To save on delineating subcases, we simply normalize this lower bound on the number of investors to one.

Although the signal space is continuous with no probability mass points, it can be used to represent discrete signals. For example, suppose investors can get either a high or a low signal, where the probability of receiving the high signal is  $q_H$  when the type is good and  $q_B < q_H$  when the type is bad. We can represent this structure in the following way:

**Example 1:** Let  $f_B(s) = 2 \times (1 - q_B)$  for  $s \in [0, 1/2]$  and  $2 \times q_B$  for  $s > 1/2$ , and let  $f_G(s) = 2 \times (1 - q_G)$  for  $s \in [0, 1/2]$  and  $2 \times q_G$  for  $s > 1/2$ , where  $0 < q_B < q_H < 1$ .

Because the likelihood ratio  $f_G(s)/f_B(s)$  is constant over  $[0, 1/2]$  and over  $(1/2, 1]$ , all signals within one of these intervals are informationally equivalent. Following [Pendorfer and Swinkels \(1997\)](#), we call such intervals “equivalence intervals.” Representing discrete signals as equivalence intervals is a convenient way of making strategies pure when they are mixed in the discrete space: one can think of a continuous signal  $s$  as a combination of a discrete signal and a random draw from the equivalence interval, where a different draw can result in a different bid even when the underlying discrete signal is the same.



## 1.1. Auction formats

Investors compete in an auction for the right to finance the project. Our main analysis is done under the assumption that the entrepreneur can only accept financing from one investor, so we model competition using one of the three standard single-unit auction formats: First-price, second-price, and ascending-price auctions.<sup>4</sup>

Our results hold both for *cash* auctions, in which investors submit cash bids for the right to take ownership of the whole project, and *security* auctions, in which investors finance the project in exchange for part of the profits in the high cash-flow state. One example of a security auction is a setting where banks offer loans at interest rate  $R_i \in [0, X]$  and the bank who submits the lowest interest rate gets to finance the project, while another is a setting where venture capitalists offer to finance the project in exchange for an equity stake.<sup>5</sup> Although the real-world applications we have in mind are usually security auctions, we choose to focus on cash auctions to make the exposition as transparent as possible and to simplify comparison with the standard auction literature. We show that all results hold for security auctions in Section 5.1.

In a first-price auction, investors submit sealed cash bids for ownership of the project. The highest bidder wins the auction and pays his bid to the seller, whereafter he decides whether to start the project or not. A second-price auction is the same except the winning investor pays the bid of the runner-up.

An ascending-price auction proceeds as follows. Bidding starts at 0 and the price is gradually increased until all but one investor remains. All bidders can see at which price other bidders drop out, and a bidder who has dropped out cannot reenter the auction. The last remaining investor wins the auction and pays the price at which the runner-up dropped out. We pay special attention to the ascending-price auction for two reasons. First, it is probably the best approximation to most real-world settings, be it formal auction procedures or informal rounds of bidding where bidders have the chance to react to competitors. Second, it has been shown to have the best information aggregation properties of all standard auctions (including multi-unit auctions; see [Kremer \(2002\)](#) and [Han and Shum \(2004\)](#)), as well as generating the highest revenues to the seller (see [Milgrom and Weber \(1982\)](#) for revenue comparisons between standard auction formats and [Lopomo \(2000\)](#) for a mechanism-design approach.) Thus, our results about the failure of information aggregation are the starkest for the ascending-price auction.

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<sup>4</sup>We discuss  $K$ -unit auctions, in which several investors can win a stake in the project simultaneously, in Section 4.3. Our result on the failure of information aggregation holds as long as the number of investors receiving an allocation remains finite as the number of potential investors grows large.

<sup>5</sup>These two examples are equivalent in our setting, since there is no distinction between debt and equity when there is only one strictly positive cash-flow state.



## 2. Equilibrium bidding

In keeping with much of the auction literature, we will be focusing on symmetric, monotone equilibria. A monotone equilibrium is one in which a bidder with a higher signal bids weakly higher (or, in the case of the ascending-price auction, drops out at a weakly higher price). With slight abuse of language, we say that bids are *strictly increasing* when they are strictly increasing except potentially over equivalence intervals.

Below, we denote the order statistics of  $N$  signals by  $Y_{1,N}, \dots, Y_{N,N}$  so that  $Y_{1,N}$  represents the highest signal,  $Y_{2,N}$  represents the second-highest signal, et cetera.

### 2.1. Equilibria without investment

The only difference between our setting and the standard auction-theory setting in [Milgrom and Weber \(1982\)](#) is that in the standard setting the object for sale is an existing asset which requires no extra financing, whereas we assume that the winner makes a decision about whether to start the project or not. As a benchmark, we start by summarizing the informational properties of auctions in the standard setting. For this purpose, assume that the investment into the project has already been made, whereafter the project is sold in an auction. Thus, the auction is of an asset that pays  $1 + X$  for good types and 0 for bad types. Symmetric equilibrium bidding strategies for the first-price, second-price, and ascending-price auctions are then as described in [Milgrom and Weber \(1982\)](#).<sup>6</sup> In particular, bids are strictly increasing in the signal of bidders. Since bids are strictly increasing, anyone who observes the history of bids ex post can recover all the information available in the market. For the ascending price auction, observing the bids is not even necessary: the equilibrium price will itself be a sufficient statistic for market information about the type (see [Kremer \(2002\)](#)). Thus, the auction generates all relevant information possessed by the bidders about the project. As a consequence, the larger the number of bidders, the more precise the information generated by the auction. As the number of bidders goes to infinity, [Assumption 1](#) ensures that the information generated by the auction perfectly reveals the type.

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<sup>6</sup>In the standard setting, [McAdams \(2007\)](#) shows that the equilibrium described in [Milgrom and Weber \(1982\)](#) is in fact the unique symmetric monotone equilibrium in the first-price auction. [Pendorfer and Swinkels \(1997\)](#) obtain a similar result for common value second-price auctions. While there is a continuum of symmetric separating equilibria in the ascending-price auction [Bikhchandani, Haile and Riley \(2002\)](#) show that they all result in a unique price.

## 2.2. Equilibria with investment

Now assume the investment is not sunk, but that the winner makes the investment decision after the auction. In Section 4.1 we analyze the situation where bids are revealed to the winner after the auction but before the investment decision, but for now we assume bids are not revealed. None of the equilibria we identify in this section are affected by the revelation of bids.

The crucial difference to the standard setting is that investors with signals below some sufficiently low cut-off  $\hat{s}$  will always elect not to start the project if they win, since winning implies that other investors also had low signals. These investors bid zero in the first-price and second-price auction, and drop out at zero in the ascending-price auction. We call the signal range  $[0, \hat{s}]$  the *informational black hole* since bids from investors in this region convey no information about the underlying signal. This information destruction of pessimistic signals is what drives all our results.

A crucial part of the analysis is therefore to determine the cut-off  $\hat{s}$ , which we call the *black-out level*. We start by deriving an upper bound. A necessary condition for  $\hat{s}$  to be an equilibrium black-out level is that a bidder with signal  $\hat{s}$  finds it optimal not to start the project when winning:

$$E(V|Y_{1,N} = \hat{s}) \leq 0. \quad (1)$$

The left-hand side of Condition 1 is the expected net present value of the project for a bidder with signal  $\hat{s}$  conditional on winning the auction; winning implies that all other signals are no higher than  $\hat{s}$ . If Condition 1 is violated, a bidder with signal  $\hat{s}$  would be strictly better off by bidding some  $\varepsilon > 0$ .<sup>7</sup>

Denote by  $\bar{s}_N$  the highest signal for which Condition 1 is satisfied, that is, using Bayes' Law, the highest signal such that

$$E(V|Y_{1,N} = \bar{s}_N) \leq 0. \quad (2)$$

Assumption 2 ensures that  $\bar{s}_N > 0$ . There is an interior solution  $\bar{s}_N \in (0, 1)$  if and only if  $E(V|S_i = 1) > 0$ , that is, if the project is positive NPV conditional on the highest possible signal realization. Otherwise, we set  $\bar{s}_N = 1$ .

We now show that  $\bar{s}_N$  can be sustained as an equilibrium in all auction formats.

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<sup>7</sup>To see this, note that if an investor with signal  $\hat{s}$  bids zero he wins the auction with probability  $\Pr(Y_{2,N} \leq \hat{s}|Y_{1,N} = \hat{s})/N$ , that is, the probability that all other bidders also bid zero and bidder 1 is allocated the project in a lottery between the tying bidders. Increasing the bid infinitesimally ensures that the probability of winning jumps up by a factor of  $N$ , and is a strictly profitable deviation if the NPV of the project conditional on winning is strictly positive.

To see this, imagine that a bidder with a signal below  $\bar{s}_N$  attempts a deviation by putting in a strictly positive bid (or, in the case of the ascending-price auction, staying in the auction at strictly positive prices). If this bid is so low that the bidder never wins unless he has the highest signal, the project is never positive NPV conditional on winning (by the definition of  $\bar{s}_N$ ), so the deviation at best generates zero profit. If the bid is high enough to beat some bidders with higher signals, the project may be positive NPV conditional on winning, but as we show in the proof of Proposition 1 below, the price paid will be higher than the value of the project.

For the first-price auction,  $\bar{s}_N$  is also the unique black-out level. To see this, suppose to the contrary that there exists an equilibrium black-out level  $\hat{s} < \bar{s}_N$ , so that a bidder with signal  $s \in (\hat{s}, \bar{s}_N)$  submits a strictly positive bid. But conditional on winning, the project is then negative NPV by the definition of  $\bar{s}_N$ , so the bidder makes strictly negative profits whether he starts the project or not.<sup>8</sup>

**PROPOSITION 1:** *In all auction formats, there is an equilibrium in which all bidders with a signal  $s \leq \bar{s}_N$  bid zero and never start the project when winning, while bidders with a signal  $s > \bar{s}_N$  submit strictly positive bids which are strictly increasing in signals, and always start the project when winning. For the first-price auction, this is the unique symmetric monotone equilibrium.*

**Proof:** See the Appendix.

Note that when  $E(V|S_i = 1) < 0$ , so that a single bidder finds it optimal not to start the project even with the most positive signal, all bidders will be in the informational black hole—the project never gets started and all information is lost. This is an inefficient outcome, since there are scenarios in which the aggregate market information would have indicated that the project is positive NPV. Hence, the auction mechanism fails completely for projects that are sufficiently “out-of-the-money”.

**Remark 1:** The equilibrium in Proposition 1 is equivalent to the equilibrium in the standard setting when there is a reserve price, as described in Section 7 of [Milgrom and Weber \(1982\)](#). To see this, imagine the following slightly modified version of our setting. There is a reserve price of 1 such that no bids below 1 are accepted. Out of the proceeds in the auction, the entrepreneur uses 1 to finance the project, while the investor who wins gets the resulting cash flows from the project. This is an auction of the project cash flows with a reserve price of 1, where the participation threshold is  $\bar{s}_N$ .

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<sup>8</sup>We show in Section 4.1 that there are other, more efficient equilibria in the first-price auction when bids are revealed after the auction.

In the equilibrium in Proposition 1, only quite optimistic investors submit non-zero bids, and when these investors win they always find it optimal to start the project. Hence, the auction mechanism does not produce any information that is useful in guiding a winner's investment decision. We now show that the second- and ascending-price auctions have other equilibria where a winner who submits a non-zero bid sometimes exercises his option of walking away from the project, resulting in a lower black-out level than  $\bar{s}_N$ . In particular, we derive a lower bound  $\underline{s}_N$  on the black-out level, given by the lowest signal solving

$$E(V|Y_{1,N} = Y_{2,N} = \underline{s}_N) \leq 0. \quad (3)$$

This is the cut-off signal above which bidders find it optimal to start the project conditional on being tied at the highest signal with one more bidder, a more favorable event than having the highest signal alone. We claim that all black-out levels between  $\underline{s}_N$  and  $\bar{s}_N$  can be supported in equilibrium. This is easiest to see for the second-price auction. Consider the following equilibrium strategies for some arbitrary black-out level  $\hat{s} \in [\underline{s}_N, \bar{s}_N]$ . Bidders below or at the black-out level bid zero. Bidders above the black-out level bid their valuation of the project conditional on just marginally winning, which happens when the second-highest bidder has the same signal:

$$b(s) = E(V|Y_{1,N} = Y_{2,N} = s).$$

When everyone follows this strategy, optimal investment behavior for a winner is as follows. If a bidder with a signal  $s$  below or at the black-out level wins the auction (at price zero), all he learns is that no bidder had a signal above  $\hat{s}$ . Since  $s \leq \hat{s} \leq \bar{s}_N$ , the project is then negative NPV and it is optimal not to invest. A bidder with a signal above the black-out level who wins the auction at a strictly positive price learns that the second-highest bidder attaches a positive value to the project when marginally winning—and this value is also the resulting price in the auction. Because the winner is more optimistic than the second-highest bidder, he finds it optimal to invest, and he makes non-negative profits at the resulting price. If he wins and the price is zero, he only learns that the second-highest bidder is below the black-out level. In this situation, he chooses to invest if and only if his own signal is above a cut-off  $\varphi(\hat{s}) > \hat{s}$  given by the highest solution to

$$E(V|Y_{1,N} = \varphi(\hat{s}), Y_{2,N} \leq \hat{s}) \leq 0. \quad (4)$$

Hence, when the winner's signal is in the interval  $(\hat{s}, \varphi(\hat{s}))$ , his investment decision

depends on what he learns in the auction, which increases the option-value of the project relative to the equilibrium in Proposition 1. It is easy to verify that this is a Nash equilibrium. Deviating with a higher bid entails winning in scenarios when the price is higher than the bidder’s expectation of the value, while deviating with a lower bid entails losing in scenarios when the price is lower than the bidder’s expectation of the value. It is also clear that there can be no equilibrium black-out level lower than  $\underline{s}_N$ , since the most favorable information a winner can learn in the auction is that the second-highest bidder has the same signal as the winner.<sup>9</sup>

The source of multiple equilibria in our setting is the negative externality on the option value of the project produced by the black hole. When the informational black hole is expected to be large, a large amount of information is destroyed and hence a winner learns comparatively little from the auction outcome. This reduces the option value of the project and hence reduces the incentive to submit non-zero bids—the expectation of a large informational black hole becomes a self-fulfilling prophecy. If the informational black hole is expected to be small, on the other hand, winners learn more in the auction, which increases the option value of the project, encouraging more investors to submit non-zero bids.

Maybe surprisingly,  $\underline{s}_N$  is also the lowest black-out level that can be sustained in the ascending-price auction, even though bidders learn more than in the second-price auction by observing when other bidders drop out. The proof is somewhat involved, so we relegate the details of the proof to the appendix and give the main arguments here. In our equilibrium construction, we have to take special care in defining how bidders can react when other bidders drop out. It is reasonable to assume that at a given price  $p$ , bidders who are still in the auction have to make simultaneous independent decisions about whether to drop out exactly at  $p$ , rather than coordinating their actions.<sup>10</sup> In our setting, this assumption can lead to situations where a bidder’s best response is to drop out just as the price goes above zero if enough other bidders drop out at zero—a strategy that is not well-defined when price is increased continuously. When such situations arise, we therefore require that the equilibrium is the limit of equilibria in auctions with discrete price increments as the size of the increment goes to zero.

With this equilibrium requirement in mind, the main argument that the black-out level cannot be below  $\underline{s}_N$  is as follows. Suppose to the contrary that there is an equilibrium in which the black-out level is some signal  $\hat{s}$  lower than  $\underline{s}_N$ , so that a bidder with a signal just slightly above  $\hat{s}$  stays in the auction until the price is slightly positive. This bidder can win under three circumstances. First, he can win if all other bidders

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<sup>9</sup>We formalize these arguments in detail in the proof of Proposition 2 in the appendix.

<sup>10</sup>This distinction is irrelevant in the standard setting of Milgrom and Weber (1982), because a bidder’s best response at  $p$  turns out to be independent of the actions of the other remaining bidders.

drop out at zero, in which case it is optimal not to start the project, which involves zero profits because the price is also zero. Second, he can win if only one other bidder stays in the auction and this bidder has a signal below  $\underline{s}_N$ , in which case it is also optimal not to start the project. Since the price is positive, this involves some losses. Third, he can win if more than two other bidders stays at positive prices, which could imply that the project is positive NPV. But in this scenario he only wins if other bidders have lower signals than him, a very small probability event. The expected profits will therefore be negative. The following proposition collects these results.

**PROPOSITION 2:** *In the second-price and ascending-price auction,  $\forall \hat{s} \in [\underline{s}_N, \bar{s}_N]$  there is a symmetric monotone equilibrium with black-out level  $\hat{s}$  and strictly increasing bidding strategies for  $s > \hat{s}$ . The project is started if and only if  $Y_{2,N} \geq \hat{s}$  or  $Y_{1,N} \geq \varphi(\hat{s})$ . If  $E(V|Y_{1,N} = Y_{2,N} = 1) \leq 0$ , the project never gets started. There are no symmetric monotone equilibria with lower black-out levels.*

**Proof:** See the Appendix.

### 3. Informational efficiency and surplus

We now turn to the main question of our paper: how efficient are auctions in allocating resources? A first-best benchmark is the surplus that can be achieved by an observer of all signals  $\mathbf{S} \equiv \{S_i : i = 1, \dots, N\}$ . We denote this surplus as  $V_N^{FB}$ , given by

$$V_N^{FB} \equiv E[\text{Max}(E(V|\mathbf{S}), 0)].$$

As  $N$  goes to infinity, there will be no investment mistakes in the first best, so  $V_N^{FB}$  goes to  $\pi X$ .

With a black-out level  $\hat{s}$ , an auction can do at most as well as an observer who sees only the censored sample  $\mathbf{S}_{>\hat{s}} \equiv \{\text{Max}(S_i, \hat{s}) : i = 1, \dots, N\}$ . We denote this surplus as  $V_N(\hat{s})$ , given by

$$V_N(\hat{s}) \equiv E[\text{Max}(E(V|\mathbf{S}_{>\hat{s}}), 0)].$$

We next show that in any auction format, an equilibrium with black-out level  $\hat{s}$  achieves exactly this surplus.

**LEMMA 1:** *An equilibrium with black-out level  $\hat{s}$  in either the first-price, second-price, or ascending-price auctions creates surplus  $V_N(\hat{s})$ .*

**Proof:** In the Appendix.

Lemma 1 shows that any equilibrium inefficiency stems purely from the size of the black-out region—given the information available from the joint set of bids, all auction formats do the best possible job. The most efficient equilibrium is one that minimizes the black-out level at  $\underline{s}_N$ , a level achievable only in the second-price and ascending-price auctions. The least efficient equilibrium is the one that maximizes the black-out level at  $\bar{s}_N$ , as in Proposition 1, which is also the unique equilibrium for the first-price auction.

Except under non-generic circumstances, any equilibrium black-out level  $\hat{s}$  leads to some inefficiency—that is,  $V_N(\hat{s}) < V_N^{FB}$ .<sup>11</sup> We now study how the surplus behaves as the number of investors with information in the market grows large. We will assume from now on that the project is positive net present value contingent on the highest signal, that is:

ASSUMPTION 3:  $E(V|S_i = 1) > 0$ .

If Assumption 3 is violated, the project never gets started in the least efficient equilibrium, even though the first-best surplus with an infinite number of bidders is  $\pi X > 0$ . We now show that there is a sizable inefficiency in the limit even if Assumption 3 holds. Adding bidders has two opposing effects on surplus. On the one hand, the aggregate amount of information in the market becomes more precise, increasing the potential surplus. On the other hand, because of the winner’s curse, the informational black hole grows with the number of bidders, so that a decreasing proportion of signals is revealed in the auction. The following lemma shows how the black-out level behaves as  $N$  grows large.

LEMMA 2: *The black-out levels  $\underline{s}_N$  and  $\bar{s}_N$  go to 1 with  $N$ :*

$$\bar{s}_N = 1 - \frac{a_1}{f_B(1)} \frac{1}{N} + o\left(\frac{1}{N}\right), \quad \underline{s}_N = 1 - \frac{a_2}{f_B(1)} \frac{1}{N} + o\left(\frac{1}{N}\right),$$

where  $a_1$  and  $a_2$  are strictly positive constants with  $a_2 > a_1$ .

*The number of bidders above the black-out level converges in distribution as  $N$  goes to infinity. In particular,*

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<sup>11</sup>The inequality is always strict if MLRP is strict. With weak MLRP and few bidders, the first best can sometimes be achieved. For example, if there are only two bidders and two equivalence intervals (as in Example 1) and if  $\hat{s} = \underline{s}_2$ , there is no inefficiency.



$$\begin{aligned}
\lim_{N \rightarrow \infty} \Pr(Y_{1,N} > \bar{s}_N | B) &= 1 - e^{-a_1}, \\
\lim_{N \rightarrow \infty} \Pr(Y_{1,N} > \bar{s}_N | G) &= 1 - e^{-\lambda a_1}, \\
\lim_{N \rightarrow \infty} \Pr(Y_{2,N} > \underline{s}_N | B) &= 1 - e^{-a_2}(1 + a_2), \\
\lim_{N \rightarrow \infty} \Pr(Y_{2,N} > \underline{s}_N | G) &= 1 - e^{-\lambda a_2}(1 + \lambda a_2).
\end{aligned}$$

**Proof:** See the Appendix.

The first part of the lemma shows that the informational black hole approaches the whole range of signals as  $N$  goes to infinity. The black-out level grows with  $N$  so that the expected number of bidders above the black-out level remains bounded and converges to a constant. For large  $N$ , the gain in aggregate information from adding a bidder is exactly balanced by the loss of one more signal into the informational black hole.

The second part of the lemma describes the probability that investments are made in the limit. The first two probabilities refer to the least efficient equilibrium, in which the project starts if and only if the highest bidder has a signal above  $\bar{s}_N$ . The lemma shows that in the limit, good projects are made with probability strictly less than one, while bad projects are made with probability strictly greater than one as long as the likelihood ratio  $\lambda$  at the top of the signal distribution is finite. Hence, the first-best is never implemented unless top signals are infinitely informative. The last two probabilities refer to the most efficient equilibrium, in which the project starts if and only if the second-highest bidder has a signal above  $\underline{s}_N$ .<sup>12</sup> Although more good investments and fewer bad investments are made in this equilibrium, the first best is still not achieved.

We next show that not only is the first best not achieved in the limit, but surplus can actually go down as more bidders are added. As Lemma 2 shows, adding more bidders does not change the expected number of observed signals, because the black-out level increases to offset the increase in the number of bidders. Observed signals will come from a higher part of the signal distribution. Whether surplus increases or decreases as the number of bidders increases depends on whether the information content of the signal distribution is concentrated towards the top. The following proposition, which gathers our results on efficiency, makes this notion precise.

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<sup>12</sup>For large enough  $N$ ,  $\varphi(\underline{s}_N) = 1$ , so investment behavior is solely determined by the signal of the second highest bidder as  $N$  gets large. We show this formally in the proof of Proposition 3.

PROPOSITION 3: *The surplus as  $N$  goes to infinity is strictly less than the first best:*

$$\lim_{N \rightarrow \infty} V_N(\bar{s}_N) \leq \lim_{N \rightarrow \infty} V_N(\underline{s}_N) < \pi X.$$

If  $\frac{f_G(s)}{F_G(s)} / \frac{f_B(s)}{F_B(s)}$  is a decreasing (increasing) function at  $s = 1$  then there is an  $\underline{N}$  such that both  $V_N(\bar{s}_N)$  and  $V_N(\underline{s}_N)$  decrease (increase) with  $N$  for  $N > \underline{N}$ .

**Proof:** See the Appendix.

The ratio  $\frac{f_G(s)}{F_G(s)} / \frac{f_B(s)}{F_B(s)}$  is a conditional likelihood ratio, which measures the informativeness of the top signal  $s$  if signals are restricted to be drawn from the interval  $[0, s]$ . If this ratio increases with  $s$ , it means that information is concentrated towards the top of the signal distribution. Adding bidders then improves efficiency, because it shifts the distribution of the pivotal order statistics  $Y_{1,N}$  and  $Y_{2,N}$  towards the top.

**Remark 2:** Any inherently discrete signal distribution has  $\frac{f_G(s)}{F_G(s)} / \frac{f_B(s)}{F_B(s)}$  decreasing in  $s$  over the highest equivalence interval, because all signals in this interval are—as the name suggests—equivalent from an informational point of view. More generally, the surplus will eventually decrease in  $N$  as long as the likelihood ratio  $f_G(s)/f_L(s)$  does not increase very much at the top of the signal distribution. On the other hand, if  $f_H(s)$  and  $f_L(s)$  are normal distributions with different means, the ratio increases with  $s$  so adding bidders improves surplus.<sup>13</sup> If  $f_G(s) = as^{a-1}$  and  $f_B(s) = bs^{b-1}$  with  $a > b$ , the ratio is constant, so the number of bidders is irrelevant for surplus.<sup>14</sup>

**Remark 3:** As we show in the proof of Proposition 3, surplus converges at a rate  $1/N$ . This implies that the effect of extra bidders on surplus, whether positive or negative, is quite large for small  $N$ , which is the empirically relevant situation in most capital raising contexts.

We use Example 1, where signals are binary, to illustrate a situation where surplus decreases with  $N$ . Suppose that  $q_B = 1/2$  and  $q_G = 1$ . Then  $f_B(s) = 1$  and  $f_G(s) = 0$  for  $s \in [0, 1/2)$  and  $f_G(s) = 2$  for  $s \in [1/2, 1]$ . Also, assume that  $\pi = 1/2$  and  $X = 1$  so that the project is zero NPV ex ante.

If there is only one bidder then the auction can stipulate any reserve price between zero and  $E(V|s \geq 1/2) = \frac{\pi X - q_B(1-\pi)}{\pi + q_B(1-\pi)} = 1/3$ . The bidder bids the reserve price if and

<sup>13</sup>The normal distribution has unbounded support, but can be represented on a unit interval by an appropriate change of variables. Note that the normal distribution also has the likelihood ratio go to infinity for top signals, so information is perfectly aggregated in the limit.

<sup>14</sup>This specification is the exponential distribution transformed to a bounded support.

only if he receives a high signal. Hence, social surplus is  $V_1 = \pi X - (1 - \pi)q_B = 1/4$ , where  $(1 - \pi)q_B$  is the probability that the project is bad and the bidder gets a high signal and invests. Note that this is equivalent to the first-best surplus with one signal.

When there are two bidders then in the most efficient equilibrium each bidder submits a nonzero bid only if he receives a high signal. The project is started only if the auction price is greater than zero. Hence, social surplus is  $V_2 = \pi X - (1 - \pi)q_B^2 = 3/8$ , where  $(1 - \pi)q_B^2$  is the probability that the project is bad and both bidders get a high signal. This is equivalent to the first-best surplus with two signals. In the least efficient equilibrium each bidder submits a nonzero bid only if he receives a signal  $s \in [\bar{s}_2, 1]$  where  $\bar{s}_2$  solves  $E(V|Y_{1,2} = \bar{s}_2) = 0$ , which using Bayes' theorem can be calculated as

$$\bar{s}_2 = \frac{1 - q_B}{1 - q_B^2 \frac{1-\pi}{\pi X}} = \frac{2}{3}.$$

If  $N > 2$ , then the blackout level  $\underline{s}_N$  in the most efficient equilibrium solves (3) and can be calculated as:

$$\underline{s}_N = \frac{1 - q_B}{1 - q_B \left( q_B^2 \frac{1-\pi}{\pi X} \right)^{\frac{1}{N-2}}}.$$

From Equation 4 one can calculate that  $\varphi(\underline{s}_N) = 1$ , so a winner never invests unless the second-highest bidder puts in a strictly positive bid, which happens when  $Y_{2,N} > \underline{s}_N$ . We can then calculate the surplus as:

$$V_N(\underline{s}_N) = \pi X \Pr(Y_{2,N} > \underline{s}_N | G) - (1 - \pi) \Pr(Y_{2,N} > \underline{s}_N | B).$$

Similarly, if  $N > 2$ , the blackout level  $\bar{s}_N$  in the least efficient equilibrium can be calculated from (2) as:

$$\bar{s}_N = \frac{1 - q_B}{1 - q_B \left( q_B \frac{1-\pi}{\pi X} \right)^{\frac{1}{N-1}}}.$$

Therefore,

$$V_N(\bar{s}_N) = \pi X \Pr(\{Y_{1,N} > \bar{s}_N\} | G) - (1 - \pi) \Pr(\{Y_{1,N} > \bar{s}_N\} | B).$$

Figure 2 plots  $V_N(\underline{s}_N)$  and  $V_N(\bar{s}_N)$  as a function of the number of bidders. In line with the results of Proposition 3 we can see that in the least efficient equilibrium social surplus declines with the number of bidders for all  $N$ . In the most efficient equilibrium maximum social surplus is achieved with just two bidders and then declines as the number of bidders increases.

### 3.1. Restricting entry and auction revenues

If the entrepreneur has the power to pick the number of bidders, he will do so in order to maximize revenues rather than surplus. The private optimum may differ from the social optimum if the entrepreneur captures only part of the surplus. In our setting, the *split* of the surplus between the entrepreneur and investors has similar comparative statics with respect to auction formats and the number of bidders as in the standard auction theory setting of [Milgrom and Weber \(1982\)](#), where surplus itself is fixed. Holding  $N$  and the black-out level fixed—so that surplus is fixed—ascending-price auctions generate higher revenues than second-price auctions, and second-price auctions generate higher revenues than first-price auctions. Also, the fraction of surplus captured by the entrepreneur goes to one with  $N$  in all auction formats. Hence, if surplus increases with  $N$ , there is no conflict between the private and social optimum—the entrepreneur will prefer the maximal number of bidders.

The non-trivial case is when surplus decreases with  $N$ . Will the entrepreneur find it optimal to restrict the number of bidders even though this may entail surrendering a higher fraction of the surplus to investors? Our answer is a qualified “Yes”. Although the privately optimal  $N$  is never lower than the social optimum, revenues are typically maximized at a finite  $N$  whenever surplus is, as we show in the following proposition:

**PROPOSITION 4:** *Suppose that there exists an  $\varepsilon > 0$  such that  $f_G(s)/f_B(s) = \lambda$  for  $s \in [1 - \varepsilon, 1]$ . Then, there exists some  $\underline{N}$  such that revenue is strictly decreasing in  $N$  for  $N \geq \underline{N}$ .*

**Proof:** We know that there exists some  $\underline{N}$  such that  $\bar{s}_N \geq 1 - \varepsilon$  for all  $N \geq \underline{N}$ . Over this interval,  $\frac{f_G(s)}{F_G(s)} / \frac{f_B(s)}{F_B(s)}$  is strictly decreasing, and so from [Proposition 3](#), surplus is decreasing in  $N$  for  $N > \underline{N}$ . All bidders must make the same expected profits since they are in the same equivalence interval. Since some bidders do not participate, the expected bidder profits are zero, and hence revenues coincide with surplus. *Q.E.D.*

To understand this result, note that surplus decreases with  $N$  when the top of the signal distribution is relatively flat, so that bidders who draw high signals are informationally close to each other. But when this is the case, bidders also capture little informational rent even for moderate levels of  $N$ . In other words, increasing  $N$  beyond a certain level has little effect on the split of revenues but a large negative effect on surplus. As an illustration, in [Example 1](#) bidders earn exactly zero surplus whenever  $N > 1$  because of competition between informationally identical bidders from the top equivalence interval. Even for  $N = 1$ , which can be viewed as a negotiation between the entrepreneur and one investor, the entrepreneur can capture the full surplus by

setting the appropriate reserve price. The plot of surplus in Figure 2 is therefore also a plot of revenues to the entrepreneur.

Our results provide one explanation for why so many capital raising situations involve negotiations with a restricted set of investors rather than an auction open to everyone.

## 4. Strategies for reducing the winner's curse

The source of inefficiency in our model is the effect the winner's curse has on the participation of pessimistic bidders, an effect that becomes stronger as the market grows larger. In this section we discuss a number of strategies that can help to alleviate the winner's curse. First, we show that revelation of bids after the auction but before the investment decision is made does not affect the inefficient investment equilibria we have described above, but can lead to more efficient equilibria in the first- and second-price auctions. Second, we show that it may be beneficial to raise capital before important information is learnt in order to increase the option value embedded in the project. Third, we show that allowing a larger set of investors to co-finance the project helps reduce the winner's curse. Finally, in contrast to results for standard auctions, we show that allowing bidders to collude ex ante via bidding clubs can also improve efficiency and revenues.

### 4.1. Ex post revelation of bids

In the first-price and second-price auction, winners do not learn the full distribution of bids from the auction itself. It is therefore natural to ask whether efficiency can be improved by revealing the bid distribution to the winner after the auction but before the investment decision.<sup>15</sup> In a standard auction where no ex post action is required, revealing the full distribution of bids would reveal all signals, and so full information aggregation would be achieved.

This is not true in our setting, because bids in the informational black hole are all equal to zero so the underlying signals cannot be recovered. Nevertheless, learning how many bidders are in the informational black hole and what the bids are outside of the informational black hole does convey extra information to the winner in the first-price and second-price auctions. Lemma 1, however, shows that this extra information has no effect on surplus, because the winner would never change his decision based on this extra information. Thus, the equilibria we establish in Proposition 2 remain equilibria

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<sup>15</sup>Revelation of bids is immaterial in the ascending-price auction because bids are observable anyway.

when bids are revealed ex post and the surplus associated with the equilibria remains unchanged.

However, the possibility of observing bids ex post creates room for the existence of other equilibria. In particular, new equilibria appear in which the first-best investment decision can be implemented as the market grows large. For any black-out level  $\hat{s}$  and any  $s, s' > \hat{s}$ , define the function

$$v(s, s'; \hat{s}) = E[\max(E[V|\mathbf{S}_{>\hat{s}}], 0) | Y_{1,N} = s, Y_{2,N} = s'].$$

to be the expectation of the project's value to the winning bidder when the signal he receives is  $s$  and the highest signal among the other bidders,  $Y_{2,N}$ , is  $s'$ . The expectation is taken over all possible signal realizations  $\mathbf{S}_{>\hat{s}}$  consistent with  $Y_{1,N} = s$  and  $Y_{2,N} = s'$ .

Define the black-out level  $s_N$  as the highest signal such that

$$E[V|Y_{1,N} = \dots = Y_{N,N} = s_N] \leq 0. \quad (5)$$

It is clear that  $s_N$  is the lowest possible black-out level. Notice that for any black-out level  $s_N < \hat{s} < \underline{s}_N$  the function  $v(s, s'; \hat{s})$  is strictly positive and goes to zero as  $s$  and  $s'$  go to  $\hat{s}$ . Hence, we can extend  $v(s, s'; \hat{s})$  to  $s = s' = \hat{s}$  by setting  $v(\hat{s}, \hat{s}; \hat{s}) = 0$ .

Having constructed the function  $v(s, s'; \hat{s})$  we can follow [Milgrom and Weber \(1982\)](#) to construct equilibrium bidding strategies in the first-price and second-price auctions. Specifically, define the bidding strategy of an investor with a signal  $s > \hat{s}$  in the second-price auction as

$$b^{II}(s) = v(s, s; \hat{s}), \quad (6)$$

and in the first-price auction as

$$b^I(s) = \int_{\hat{s}}^s v(s', s'; \hat{s}) dL(s'|s), \quad (7)$$

where

$$L(s'|s) = \exp\left(\int_s^{s'} \frac{h(s'|s)}{H(s'|s)} dt\right).$$

The function  $H(\cdot|s)$  is the distribution of  $Y_{2,N}$  conditional on  $Y_{1,N} = s$  and  $h(\cdot|s)$  is the associated conditional density function. [Proposition 5](#) establishes that bidding strategies  $b^{II}(s)$  and  $b^I(s)$  together with any black-out level  $\hat{s} \in [s_N, \bar{s}_N]$  form an equilibrium in the second-price and first-price auctions respectively.

**PROPOSITION 5:** *When bids are revealed after the auction but before the decision on whether to start the project, for any black-out level  $\hat{s} \in [s_N, \bar{s}_N]$ , bidding strategies  $b^{II}(s)$*

and  $b^I(s)$  form an equilibrium in the second-price and first-price auctions respectively.

**Proof.** See the Appendix.

Notice that as  $N$  goes to infinity,  $s_N$  converges to the value that solves  $f_G(s)/f_B(s) = 1$ . Since the expected number of signals above this black-out ratio grows without bound with  $N$ , observing those signals perfectly reveals the project's type in the limit, and so first-best efficiency can be restored.

The result in Proposition 5 shows that first-price and second-price auctions have equilibria that are more efficient and generate more revenues than any equilibrium of the ascending-price auction. Thus, the revenue-ranking in the standard setting, where the ascending-price auction generates most revenues, can be reversed in our setting.

## 4.2. Timing of the auction and the linkage principle

Suppose that there is some exogenous signal affiliated with the value of the project that gets realized either before or after the auction. For example, this could be a signal about demand conditions for the products the project is meant to create, or any information the entrepreneur might have about the project that can be credibly communicated to the bidders. The question we ask is whether it is better to run the auction before or after this information is released.

For standard auctions, where no action is taken, the linkage principle of [Milgrom and Weber \(1982\)](#) suggests that it is better to run the auction after all value-relevant information is realized in order to lower the informational asymmetry between bidders. However, in our setting we have an extra effect: If the signal is revealed after the auction but before the investment decision is made, the project has some real option value when bids are submitted, and so even bidders with low signals might want to participate. This could break the destruction of information.

We now give an example where the linkage principle fails in our setting. Suppose that a public signal  $S_P \in \{s_G, s_B\}$  will be released at date  $t$ , where  $\Pr(S_P = s_G|B) = 0$  and  $\Pr(S_P = s_G|G) = q$ ,  $q \in (0, 1)$ . Hence, when the public signal is  $s_G$ , the project NPV is positive regardless of the bidders' signals.

Suppose first that the entrepreneur runs the auction after the public information is released, as the linkage principle prescribes. We now calculate the expected surplus generated by the auction. With probability  $q\pi$  the public signal reveals that the project is good, so surplus is  $X$ . With probability  $(1 - q)\pi + 1 - \pi$ , the public signal is  $s_B$  and the updated prior on the project being good is  $\hat{\pi} = \frac{\pi(1-q)}{\pi(1-q)+(1-\pi)} < \pi$ , in which case the auction generates some surplus  $W$ , which from Proposition 3 is strictly below the



first-best surplus  $\hat{\pi}X$ . The expected surplus is then

$$q\pi X + ((1 - q)\pi + 1 - \pi)W < \pi X.$$

Suppose to the contrary that the entrepreneur runs the auction before the public signal is released, and that winners can wait to observe the public signal before they make the decision to start the project. In this case, everyone participates in the auction and there is no informational black hole. To see this, notice that even for the most pessimistic bidders, the option to do the project has some strictly positive value since there is always some strictly positive probability that the public signal will reveal the project to be good. It is then easy to verify that bids will be strictly positive and strictly increasing in signals for all  $N$ . As a result, all informational properties of the auction are the same as in the standard setting. In particular, ascending-price auctions aggregates all information and leads to first-best investment decisions when the market grows large, and the same holds for first-price and second-price auction if bids are revealed ex post. Furthermore, the expected revenue converges to the expected surplus as  $N$  goes to infinity. Hence, the seller is better off running the auction before the public signal is revealed.

**Remark 4:** Our exercise in this section compares the effect of running the auction before or after some public release of information, rather than asking whether releasing information is better than never releasing it at all. In the standard model of [Milgrom and Weber \(1982\)](#) this distinction is irrelevant, since ex post releases of information have no impact on the expected value of the asset up for sale. If the choice is whether to release information before the auction or never, Theorem 18 of [Milgrom and Weber \(1982\)](#) can be applied to show that the linkage principle holds for the least efficient equilibria described in [1](#). Whether this version of the linkage principle holds for the wider set of equilibria in [Proposition 2](#) is an open question.

**Remark 5:** The results in this section show that if the decision to start the project can be postponed indefinitely and costlessly, and if there is any possibility that the project can become positive net present value sometime in the future even for the most pessimistic investors, then the black hole will be eliminated and the auction will properly aggregate information (assuming bids are revealed ex post). Hence an important underlying assumption for our results is that the option to start the project has some natural expiration date, or that there are sufficient costs associated with keeping the option alive. We believe this to be a natural assumption for most real options.

### 4.3. K-unit auctions

In the previous sections we assumed that only one investor can finance the project ex post. In this section we allow for the possibility that  $K > 1$  investors can co-finance the project. Allowing for more investors to receive an allocation in the auction weakens the winner’s curse and hence encourages more investors to submit non-zero bids, which has a positive effect on efficiency. [Pesendorfer and Swinkels \(1997\)](#) show that the  $K$ -unit auction has a unique symmetric monotone equilibrium in the standard setting and that the auction fully aggregates information as  $N \rightarrow \infty$  if and only if  $K$  satisfies the “double largeness” condition:  $K \rightarrow \infty$  and  $N - K \rightarrow \infty$ .

While there are multiple equilibria in our setting, we show that the aggregation properties of  $K$ -unit auction mirror those of [Pesendorfer and Swinkels \(1997\)](#). In particular, inefficiencies persist as long as  $K$  is finite, even if the bids are made known after the auction and are incorporated in the investment decision. The case of finite  $K$  seems reasonable in most corporate finance situations. If  $K$  is allowed to grow proportionately with  $N$ , then inefficiencies disappear in the limit.

Specifically, we assume that the  $K$  highest bidders who submit nonzero bids share equally the investment costs and the project’s payoff. Each bidder pays the bid submitted by the  $K + 1^{st}$  highest bidder. If there are less than  $K$  bidders who submit nonzero bids the project is cancelled. Otherwise the  $K$  highest bidders get the right to finance the project. In principle, winning bidders may disagree about the decision to start the project. When  $K$  grows with  $N$  we show that for large  $N$  all winning bidders agree on the investment decision. When  $K$  is finite we consider the optimistic scenario in which all winning investors share their information with each other and jointly decide whether to start the project.

**PROPOSITION 6:** *In the  $K$ -unit auction, for any finite  $K$ , the limiting surplus is strictly lower than the first-best expected surplus  $\pi X$ . If  $K/N$  goes to some constant larger than zero and smaller than one, then the expected surplus converges to the first-best expected surplus.*

**Proof:** See the Appendix.

Our results in this section can be used to explain why firms explicitly ration the allocation of shares in initial public offerings so that a larger number of investors receive an allocation. It can also explain why entrepreneurs often allow a number of venture capitalists to co-invest.

**Remark 6:** [Atakan and Ekmekci \(2014\)](#) study  $K$ -unit auctions in which double-largeness holds and in which information is not fully aggregated in the limit. Their

equilibria are specific to the multi-unit setting and fail to exist in a single-unit setting. Our results are the reverse—information is aggregated when double-largeness holds but not when  $K$  is finite. In this sense, our papers are complementary. Another important difference is that the non-revealing equilibria in [Atakan and Ekmekci \(2014\)](#) require some winning bidders to take a different action than others after the auction, while we assume that winners have to take a joint action (start the project or not). Our assumptions are appropriate in a project financing context, while the assumptions in [Atakan and Ekmekci \(2014\)](#) are better fitted to situations such as the sale of a number of plots of land that can be put to different use by different owners.

#### 4.4. Syndicates and club bids

We now study a setting in which bidders can form consortia and submit a joint bid. We provide an example in which allowing such “club bids” has a positive effect on surplus and revenues. This is in contrast to the intuition from the standard setting, where collusion among bidders tends to lower seller revenues.

Assume there are  $N \times M$  bidders in the auction, and that the signal distribution is as in [Example 1](#). We will contrast two market settings. In the first, there is no collusion among bidders and everyone submits bids independently, so that the upper bound on surplus and revenues is  $V_{N \times M}(\underline{s}_{N \times M})$ . In the second, investors are randomly allocated to  $N$  symmetric clubs each consisting of  $M$  investors, whereupon each club submits a joint bid in the auction. Our question is whether an auction with club bids generates more revenue than a non-collusive auction as  $N \times M$  grows large.

A sufficient statistic for the signals of the members in each club is then the sum of the individual signals, which follows a multinomial distribution where the likelihood ratio  $\lambda_M$  at the top of the signal distribution is given by

$$\lambda_M = \left( \frac{q_G}{q_B} \right)^M .$$

As a benchmark, we first consider the standard auction setting where the asset for sale is already in place. In this setting, surplus is always the same. Under the assumptions of [Example 1](#), the results in [Axelson \(2008\)](#) imply that in the first-price and second-price auctions, larger clubs lead to lower revenues when the number of participants is large.

In our investment setting, suppose we hold  $M$  fixed and let  $N \times M$  grow large so that the number of clubs  $N$  grows large. Parameters other than  $\lambda_M$  determining asymptotic surplus remain fixed as  $M$  varies. Since the asymptotic surplus increases

in  $\lambda$ , it then follows that for large enough  $N \times M$ , surplus is higher if the club is bigger. Also, since the total surplus goes to the entrepreneur as  $N \times M$  grows large, the entrepreneur is better off with club bids.

A slightly different thought experiment is one in which  $M$  grows large, holding  $N$  fixed. The information held by each club then converge to the truth as  $M$  goes to infinity. It is easy to see that this implies that the first-best surplus  $\pi X$  will be reached in the limit, and that this surplus will go to the entrepreneur.

To summarize, club bids increase entrepreneurial revenues in situations where there are either sufficiently many clubs, or sufficiently many participants in each club. Our theory provides a benign rationale for the prevalent use of club bids in private equity and the use of syndicates in venture capital that has come under scrutiny by competition authorities.

**Remark 7:** For simplicity, we have assumed that clubs are symmetric and that the allocation to clubs is exogenously given, rather than allowing for endogenous club formation and for the possibility of clubs of different sizes bidding against each other. A fuller treatment would require that we abandon our focus on symmetric equilibria, and is beyond the scope of our paper.

## 5. Robustness of the model

In this section, we discuss a number of robustness issues related to the auction format, the assumption of zero wealth for the firm, and the equilibrium selection of our model.

### 5.1. Security auctions

We first show that all our results remain true in the case of security auctions, in which investors finance the project in exchange for part of the profits. Because the project's payoff in our setting is either 0 or  $1 + X$  a security auction takes a particularly simple form: bidders submit interest rate  $R_i \in [0, X]$  at which they are willing to finance the project. The auction proceeds in the same way as for cash auctions, except that the winner is the bidder submitting the lowest interest rate (or, in the case of the ascending-price auction, the final remaining bidder as the interest rate is gradually lowered by the auctioneer.)

We assume that the decision to start the project rests with the entrepreneur unless the winning bid is  $X$ , in which case the entrepreneur gives up all the cash flow rights, and therefore control rights are transferred to the winning bidder.

Notice that whenever the winning bid is below  $X$  the entrepreneur always start the project. Hence, a bidder who submits a bid below  $X$  should be prepared to finance and start the project if he wins the auction. Hence, the black-out region in the first-price security auction is exactly the same as the one in the first-price cash auction.

In the second-price and ascending-price security auctions, a winner who gets to finance the project at the interest rate  $X$  has an option not to start it even if his own winning bid is below  $X$ . This is the same option that a winner in the cash option has when the latter wins and pays 0. Thus, there is one-to-one map between the size of the informational black hole in the second-price and ascending-price security and cash auctions.

Because social surplus depends solely on the size of the informational black hole, social surplus is the same in the security auction as in the cash auction.

## 5.2. Assets in place and entrepreneurial wealth

We have assumed that the entrepreneur has no wealth of his own to finance the project, and no other assets that can be pledged to investors in exchange for financing. The model easily extends to the case of an existing firm raising financing for a new project, where the firm could either use some of its cash to co-finance the project or issue securities that are backed not only by the cash flows of the new project but also by the existing assets of the firm.

First, imagine that the entrepreneur has some wealth  $w$ , and issues an equity stake backed by a fraction  $1 - w$  of the cash-flows of the project, where the winner invests  $1 - w$  and the entrepreneur invests  $w$  to start the project if they find it optimal to do so. It is easy to see that this leads to the exact same equilibria as when there is no wealth, except that all prices and bids are scaled down by a factor  $1 - w$ . Hence, surplus is exactly the same independent of the wealth of the entrepreneur. The only change is that revenues of the entrepreneur go up with wealth, since the fraction of surplus captured by investors goes down by a factor  $1 - w$ . This effect reinforces our result in Proposition 4 that revenues can go down with the number of bidders: as  $w$  goes to one, revenues will behave in exactly the same way as surplus.

One can also show that the entrepreneur would never want to subsidize investors by giving up a larger share of the project than  $1 - w$ . Doing so would lower equilibrium black-out levels, but only because investors sometimes would find it optimal to pursue negative NPV projects, which would lead to a destruction of surplus.

Now suppose that the entrepreneur does not have liquid wealth, but has an existing firm with assets that can be pledged to back the security issue. For example, suppose

the firm has assets in place with random but positive cash flows  $Z$  uncorrelated with the project's cash flows and that the firm issues new shares backed by both the assets in place and the new project. Suppose the firm runs a security auction in which investors bid the fraction of shares  $\alpha$  they are willing to accept in exchange for the capital needed to finance the project. The most pessimistic investors would then submit a bid of  $1/E(Z + 1)$ ; this is the fraction of shares needed to break even on an investment of 1 if the project is not pursued and the money raised is kept within the firm. The equilibrium black-out level below which investors submit this bid would be exactly the same as in our original model, so surplus would also remain the same. Again, as in the case of wealth, the entrepreneur would capture a larger share of the surplus the larger the value of the existing assets are, but investment efficiency would not be improved.

### 5.3. Equilibrium robustness and alternative mechanisms

Our goal in this paper is to analyze how standard selling mechanism solve the information aggregation problem, rather than deriving an optimal procedure using a mechanism design approach. It is well-know that in a pure common value setting such as ours, there are mechanisms that can fully extract the information of bidders at virtually no cost for the entrepreneur (see for example [Cremer and McLean \(1985\)](#)). These mechanisms typically look esoteric relative to real-world auctions and involve features that are non-robust to some reasonable restrictions, such as requiring that losers should not have to pay or assuming that the mechanism is ex-post implementable in the sense that bidders should not feel regret about their strategy after the mechanism is run.

One reasonably simple mechanism that would satisfy standard robustness criteria is to allow a winner not to pay his bid if the project is not started. The equilibria we have identified in [Proposition 2](#) would not change in such a setting. However, this mechanism would feature extra equilibria, some of which are more efficient. For example, the ascending-price auction would have an equilibrium much like the ones we identified for first-price and second-price auctions when bids are revealed ex post, in which the first-best can be achieved as the market grows large. There would also exist a completely inefficient equilibrium in which everyone stays in the auction up to some maximum price at which no bidder would like to invest. One can show that none of these extra equilibria are robust to assuming a very small but strictly positive cost of submitting a bid, which leads us to the topic of the next section: how robust are our equilibria to small perturbations of the model?

We have assumed that investors always put in a bid of at least zero. If we instead

allowed investors below the black-out level to simply not show up for the auction, one can show that the ascending-price auction can have extra equilibria in which the first-best is implemented as the market grows large. The reason is that the bidders who do show up learn immediately how many bidders are below the black-out level and therefore can make their decision on whether to stay in the auction at strictly positive prices contingent on this information. This in turn helps support lower, more efficient black-out levels because more pessimistic bidders are willing to participate.

However, these equilibria are knife-edge because they rely on bidders being exactly indifferent between participating or not around the black-out level. Any minimal participation cost or any minimal positive value of participating (such as the situation when some minimal amount of assets-in-place also back the issue) will destroy these equilibria. In fact, one can show that a small participation cost will also destroy all but the least efficient equilibrium in Proposition 1 would remain.

Similarly, the efficient equilibria we have identified for the first-price and second-price auctions when bids are revealed ex post are somewhat fragile because they require that all bidders are absolutely certain that other bidders coordinate on the same equilibrium. If we required that bidders had to put some arbitrarily small probability on other bidders coordinating on another equilibrium, one can show that only the equilibria in Proposition 2 remain. In summary, the main message of our paper—that the standard auction mechanisms fail in aggregating information and lead to substantial investment inefficiencies—appears robust.

## 6. Conclusion

Our paper points to the detrimental effect of the winner’s curse on information aggregation in the important setting of project financing. Ignoring this effect leads to an overly optimistic view of the capability of financial markets to allocate resources efficiently. Our analysis also shows that several intuitive prescriptions from standard auction theory need to be reexamined when information has a real allocational role: a more competitive market is not always better, early releases of information may be suboptimal, and collusion among bidders may be beneficial to the seller.



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## Appendix. Proofs

**Proof of Proposition 1:** The text shows that  $\bar{s}_N$  is an upper bound on the black-out level, and that there can be no lower black-out level in the first-price auction. We now prove that equilibria with this black-out level exist for all auction formats. Assume that the black-out level  $\bar{s}_N$  is part of a bidding equilibrium. From the definition of  $\bar{s}_N$ , it follows that anyone who wins with a signal above  $\bar{s}_N$  finds it optimal to start the project, regardless of what bids are observed in the auction. As we show in Remark 1, if a bidder who submits a non-zero bid is forced to start the project, our setting is equivalent to the setting in Milgrom and Weber (1982) with a reserve price. Milgrom and Weber (1982) then prove existence for an equilibrium with black-out level  $\bar{s}_N$  in all auction formats. What remains to be shown is that no bidder with a signal below  $\bar{s}_N$  has an incentive to deviate by submitting a non-zero bid, and under some circumstances not start the project when winning. We have already shown in the text that no such deviation is possible in the first-price auction. We now show that this is true also for the other auction formats. We denote the net present value of the project if started by  $v$ .

*Second-price auctions:* In the second-price auction, equilibrium bids in Milgrom and Weber (1982) for bidders with signals above the black-out level are given by

$$b(s) = E(v|Y_{1,N} = Y_{2,N} = s).$$

A bidder with a signal  $S_i = s < \bar{s}_N$  who deviates and bids  $b' > 0$  will win whenever  $Y_{1,N} < b^{-1}(b')$ . From the definition of  $\bar{s}_N$ , this bidder will not start the project conditional on winning when the price is zero, and so will make zero profits in this scenario. When the price is  $p > 0$ , he loses money if he does not start the project, and his expected profits from starting the project are

$$E(v|Y_{1,N} = b^{-1}(p), S_i = s) - p,$$

which is strictly negative since  $s < b^{-1}(p)$  and since  $p = E(v|Y_{1,N} = Y_{2,N} = b^{-1}(p))$ .

*Ascending-price auctions:* In the ascending-price auction, a bidder with a signal  $s$  above the black-out level drops out at a price  $p > 0$  which is equivalent to the conditional expectation of  $v$  given the observed drop-outs below  $p$  and given that all other remaining bidders have the same signal  $s$ . By a similar argument to the one made for the second-price auction, a bidder below the black-out level who deviates by staying in the auction at non-zero prices will then win only when his expected value of the project is strictly below the price he has to pay, and hence will make strictly

negative profits. *Q.E.D.*

**Proof of Proposition 2:**

*Second-price auctions:* The text proves that the black-out level cannot be below  $\underline{s}_N$  or above  $\bar{s}_N$ . We now construct equilibria for all black-out levels  $\hat{s} \in [\underline{s}_N, \bar{s}_N]$ . Denote the net present value of the project if started by  $v$ . Our postulated equilibrium is such that all bidders with signals below  $\hat{s}$  bid zero, while bidders with signals  $s \geq \hat{s}$  bid  $b(s)$  given by

$$b(s) = E(v|Y_1 = Y_2 = s).$$

It is easy to verify that  $b(s)$  is strictly increasing (except if  $N = 2$  and  $s$  is in an equivalence interval, where it is also easy to verify that this is an equilibrium). Given this postulated equilibrium bid function, a bidder who wins the auction at price zero values the project at  $E(v|Y_1 = s, Y_2 \leq \hat{s})$ , which is negative if and only if  $s < \bar{s}_N(\hat{s})$ . A bidder who wins the auction at price  $p > 0$  values the project at  $E(v|Y_1 = s, Y_2 = b^{-1}(p))$ , which is negative if and only if  $s \leq \hat{s}$ . This confirms that equilibrium investment behavior is as described in the proposition. To confirm that this constitutes a Nash equilibrium, suppose a bidder with signal  $s$  deviates by bidding  $b' > b(s)$ . Profits are unchanged if the price is below  $b(s)$  or above  $b'$ . Otherwise, profits are  $\max[E(v|S_i = s, Y_1 = b^{-1}(p)), 0]$ , which is non-positive since  $s < b^{-1}(p)$ . A similar argument shows that deviating to a lower bid does not increase profits.

*Ascending-price auctions:* The text proves that the black-out level cannot be above  $\bar{s}_N$ . We start by verifying that equilibria as in the proposition exist, then prove that there can be no equilibrium with a black-out level below  $\underline{s}_N$ .

Suppose a black-out level  $\hat{s} \in [\underline{s}_N, \bar{s}_N]$ . We construct an equilibrium following the strategy in [Milgrom and Weber \(1982\)](#) modified to our setting. Bidders with signals below or at the black-out level drop out at  $p = 0$ . A strategy for a bidder with signal  $s > \hat{s}$  who has not dropped out at  $p = 0$  is described by functions  $b_{k,m}(s|p_1, \dots, p_k; m)$  which specify the price at which the bidder will quit if  $m$  bidders have dropped out at zero and  $k$  bidders have dropped out at non-zero prices  $p_1 \leq \dots \leq p_k$ . We need to define these functions only for  $m < N - 1$ , because otherwise the auction stops immediately and the winner pays zero. For  $m < N - 1$ , we define the strategy  $b_{k,m}$  as follows.

$$\begin{aligned} b_{0,m}(s|m) &= \text{Max}(0, E[V|Y_{m,N} \leq \hat{s}, Y_{1,N} = s, \dots, Y_{n-m-1,N} = s]), \\ b_{k,m}(s|p_1, \dots, p_k; m) &= E[V|Y_{m,N} \leq \hat{s}, Y_{1,N} = s, \dots, Y_{N-m-k-1,N} = s, \\ &= b_{k-1,m}(Y_{n-m-k}|p_1, \dots, p_{k-1}; m) = p_k, \dots, b_{0,m}(Y_{N-m-1}|m) = p_1]. \end{aligned}$$

Notice that from the definition of  $\underline{s}_N$ ,  $b_{0,m}$  is strictly greater than zero. Also,  $b_{k,m} > p_k$  for  $k > 0$ . Hence, the strategies satisfy our condition that bidders have to make their drop-out decision at price  $p$  independently of other remaining bidders.

The proof that the above strategies are indeed the equilibrium strategies for investors with signals above the black-out level is then the same as in [Milgrom and Weber \(1982\)](#). Thus, we only need to verify that any investor with a signal below  $\hat{s}$  will not deviate. Suppose he deviates and does not drop out. If he is the only one left, then the project is negative NPV, and therefore, it should not be started. If not all investors quit at zero price then if the deviating investor wins the auction he pays  $E[V|Y_{m-1,N} \leq \hat{s}, Y_{1,N} = y_1, \dots, Y_{N-m,N} = y_{n-m}]$ , where  $y_1, \dots, y_{n-m}$  are the realizations of  $Y_{1,N}, \dots, Y_{N-m,N}$ . Therefore, his conditional expected payoff is negative, and therefore, he should not deviate in the first place.

To prove that the black-out level cannot be below  $\underline{s}_N$  in the ascending-price auction, we construct equilibria in which the price in the auction is increased by  $\delta > 0$  in each round, and take the limit as the step size goes to zero. Suppose to the contrary that there is an equilibrium with black-out level  $\hat{s}_N < \underline{s}_N$  so that any bidder with a signal  $s' > \hat{s}_N$  stays in the auction until the price reaches  $\delta$ . For a given realization of signals, let  $n$  be the number of bidders who stay in the auction. Condition (3) implies that if  $n = 2$  then in any monotone equilibrium any bidder  $s'$  with  $s' \in (\hat{s}_N, \underline{s}_N]$  should drop out at price  $\delta$ . If the other bidder also has a signal in the interval  $(\hat{s}_N, \underline{s}_N]$  then each wins the auction with probability  $1/2$  and realizes a loss  $\delta$ . Therefore, the expected loss for a bidder with signal  $s' \in (\hat{s}_N, \underline{s}_N]$  is at least  $L = \delta \times Pr(\hat{s}_N < Y_{1,N-1} \leq \underline{s}_N, Y_{2,N-1} \leq \hat{s}_N)/2$ .

Without much loss of generality we can assume that

$$E(V|Y_{1,N} = Y_{2,N} = Y_{3,N} = \hat{s}_N) > 0, \tag{A1}$$

which implies that if  $n \geq 3$  and the bidder with signal  $s'$  wins the auction then he realizes some profit. The bidder with signal  $s'$  can win the auction in two cases. First, he wins if all other bidders have a lower signal than  $s'$ . Clearly, the profit cannot be greater than  $X$  and as  $s' \rightarrow \hat{s}_N$  the probability of this event goes to zero. Thus, there exists  $\varepsilon > 0$  such that for any  $s' \in (\hat{s}_N, \hat{s}_N + \varepsilon)$  the expected gain is less than  $L/2$ . Second, because price increases are discrete,  $s'$  can win if bidders with higher signals will drop at the same price as  $s'$  does. Notice that as  $\delta$  goes to zero the probability of this event goes to zero while the maximum gain for a bidder with signal  $s'$  is no more than the price increment  $\delta$ . Therefore, there exists  $\delta > 0$  such that the expected gain is less than  $L/2$ . Thus, we have showed that for any  $s' \in (\hat{s}_N, \hat{s}_N + \varepsilon)$  the expected loss

is larger than the expected gain. Therefore,  $\hat{s}_N$  cannot be the participation threshold. *Q.E.D.*

**Proof of Lemma 1:** Define the following four mutually exclusive and collectively exhaustive sets of signal realizations:  $A1 \equiv \{\mathbf{S} : Y_{1,N} \leq \hat{s}\}$ ,  $A2 \equiv \{\mathbf{S} : Y_{2,N} \geq \hat{s}\}$ ,  $A3 \equiv \{\mathbf{S} : Y_{2,N} \leq \hat{s}, Y_{1,N} \in [\hat{s}, \varphi(\hat{s})]\}$ ,  $A4 \equiv \{\mathbf{S} : Y_{2,N} \leq \hat{s}, Y_{1,N} \geq \varphi(\hat{s})\}$ . In all equilibria, we have  $\hat{s} \in [\underline{s}_N, \bar{s}_N]$ . For such  $\hat{s}$ , given  $\mathbf{S}_{>\hat{s}}$  the optimal investment behavior is to not invest whenever  $\mathbf{S}_{>\hat{s}} \in A1 \cup A3$ , and to invest whenever  $\mathbf{S}_{>\hat{s}} \in A2 \cup A4$ , exactly as the equilibrium investment behavior in Proposition 2. *Q.E.D.*

**Proof of Lemma 2:** Equations (2) and (3) imply that  $\bar{s}_N$  and  $\underline{s}_N$  are defined by

$$\frac{F_G^{N-1}(\bar{s}_N) f_G(\bar{s}_N)}{F_B^{N-1}(\bar{s}_N) f_B(\bar{s}_N)} = \frac{1}{zX}, \quad (\text{A2})$$

$$\frac{F_G^{N-2}(\underline{s}_N) f_G^2(\underline{s}_N)}{F_B^{N-2}(\underline{s}_N) f_B^2(\underline{s}_N)} = \frac{1}{zX}. \quad (\text{A3})$$

Taking the logarithm of the both parts of the above equations we have

$$(N-1) \ln \left( \frac{F_G(\bar{s}_N)}{F_B(\bar{s}_N)} \right) + \ln \left( \frac{f_G(\bar{s}_N)}{f_B(\bar{s}_N)} \right) = -\ln(zX), \quad (\text{A4})$$

$$(N-2) \ln \left( \frac{F_G(\underline{s}_N)}{F_B(\underline{s}_N)} \right) + 2 \ln \left( \frac{f_G(\underline{s}_N)}{f_B(\underline{s}_N)} \right) = -\ln(zX). \quad (\text{A5})$$

Equations (A4) and (A5) imply that both  $\bar{s}_N$  and  $\underline{s}_N$  go to one as  $N$  goes to infinity. Taking Taylor series of (A4) and (A5) and using that

$$\begin{aligned} \lim_{s \rightarrow 1} F_G(s) &= 1 - f_G(1)(1-s), \\ \lim_{s \rightarrow 1} F_B(s) &= 1 - f_B(1)(1-s), \\ \lim_{s \rightarrow 1} \frac{f_G(s)}{f_B(s)} &= \lambda, \end{aligned}$$

we obtain that

$$1 - \bar{s}_N = \frac{a_1}{f_B(1)} \frac{1}{N} + o(1/N), \quad a_1 = \frac{\ln(\lambda zX)}{\lambda - 1}, \quad (\text{A6})$$

$$1 - \underline{s}_N = \frac{a_2}{f_B(1)} \frac{1}{N} + o(1/N), \quad a_2 = \frac{\ln(\lambda^2 zX)}{\lambda - 1}. \quad (\text{A7})$$

The lemma's statements then follow from Theorem 4.2.3 of Embrechts, Klüppelberg and Mikosch (2012). *Q.E.D.*

**Proof of Proposition 3:** We only need to prove the comparative statics results with respect to  $N$ . To simplify the derivations we renormalize the densities  $f_B$  and  $f_G$  so that  $f_B(1) \equiv 1$  and  $f_G(1) = \lambda$ . Taking Taylor series of (A4) and (A5) we obtain the following results

$$1 - \bar{s}_N = \frac{a_1}{N} + \frac{b}{N^2} + o(1/N^2), \quad , \quad (\text{A8})$$

$$1 - \underline{s}_N = \frac{a_2}{N} + \frac{b}{N^2} + o(1/N^2), \quad (\text{A9})$$

where  $a_1$  and  $a_2$  are given by (A6) and (A7) respectively, and

$$b = \frac{\lambda a^2(f - \lambda(\lambda - 1)) - 4af}{2\lambda(\lambda - 1)}, \quad f = f'_G(1). \quad (\text{A10})$$

In the least efficient equilibrium social surplus is

$$V_N(s_N^*) = \pi X \Pr(Y_{1,N} > s_N^*|G) - (1-\pi) \Pr(Y_{1,N} > s_N^*|B) = \pi X F_G^N(s_N^*) - (1-\pi) F_B^N(s_N^*). \quad (\text{A11})$$

Substituting (A8) into (A11) we obtain the following expression for the surplus

$$\begin{aligned} V_N(s_N^*) &= \pi X - (1-\pi) \left( 1 - (\lambda z X)^{-\frac{1}{\lambda-1}} \left( 1 - \frac{1}{\lambda} \right) \right) \\ &\quad + (1-\pi) (\lambda z X)^{-\frac{1}{\lambda-1}} \frac{a_1^2(\lambda(\lambda-1) - f)}{2\lambda N} + o(1/N). \end{aligned} \quad (\text{A12})$$

Equation (4) implies that  $\varphi$  is defined by

$$\frac{F_G^{N-1}(\underline{s}_N) f_G(\varphi)}{F_B^{N-1}(\underline{s}_N) f_B(\varphi)} = \frac{1}{zX}$$

if

$$\frac{F_G^{N-1}(\underline{s}_N)}{F_B^{N-1}(\underline{s}_N)} \geq \frac{1}{\lambda X}, \quad (\text{A13})$$

and is equal to 1 otherwise. Using (A3) we can write condition (A13) as

$$\frac{F_G(\underline{s}_N)}{F_B(\underline{s}_N)} \geq \frac{1}{\lambda} \frac{f_G^2(\underline{s}_N)}{f_B^2(\underline{s}_N)}. \quad (\text{A14})$$

As  $N$  goes to infinity, the LHS of (A14) is bounded by one, while the RHS of (A14) goes to  $\lambda > 1$ . Thus, inequality (A14) does not hold. Hence, for  $N$  sufficiently large

$\varphi = 1$ . Therefore, social surplus is given by

$$V_N(\underline{s}_N) = \pi X \Pr(Y_{2,N} > \underline{s}_N | G) - (1 - \pi) \Pr(Y_{2,N} > \underline{s}_N | B).$$

Notice that

$$\Pr(Y_{2,N} > s) = 1 - NF^{N-1}(s) + (N-1)F^N(s). \quad (\text{A15})$$

Substituting (A9) into (A15) we obtain the following expression for the surplus

$$\begin{aligned} V_N(\underline{s}_N) = \pi X - (1 - \pi) & \left( 1 - (\lambda^2 z X)^{-\frac{1}{\lambda-1}} \left( 1 - \frac{1}{\lambda^2} + \frac{a_2(\lambda-1)}{\lambda} \right) \right) \\ & + (1 - \pi) (\lambda^2 z X)^{-\frac{1}{\lambda-1}} \frac{a_2^3(\lambda(\lambda-1) - f)}{2\lambda N} + o(1/N). \end{aligned} \quad (\text{A16})$$

Expressions (A12) and (A16) imply that both  $V_N(\bar{s}_N)$  and  $V_N(\underline{s}_N)$  decrease with  $N$  if  $f < \lambda(\lambda-1)$ . Notice that if  $f_B(s) \equiv 1$ , then  $\frac{F_G(s) f_B(s)}{F_B(s) f_G(s)} = \frac{F_G(s)}{s f_G(s)}$ . Taking the derivative of  $\frac{F_G(s)}{s f_G(s)}$  at  $s = 1$  we can see that it is positive if  $f < \lambda(\lambda-1)$  and is negative if  $f > \lambda(\lambda-1)$ . *Q.E.D.*

**Proof of Proposition 5:** Consider first investors with signals above the black-out level. For the first-price auction the proof follows the same steps as the proof of Theorem 14 of Milgrom and Weber (1982). For the second-price auction and  $\hat{s} \in [s_N, \underline{s}_N]$  the proof follows the same steps as the proof of Theorem 6 of Milgrom and Weber (1982). Proposition 2 covers the case of  $\hat{s} \in [\underline{s}_N, \bar{s}_N]$  for the second-price auctions. For investors with signals below the black-out level the proof follows the same steps as the proof of Proposition 2. *Q.E.D.*

**Proof of Proposition 6:** We first prove that the expected surplus in the  $K$ -unit auction if  $K$  is finite is strictly lower than  $\pi X$ , even if winning investors share their signals before the decision to invest is made. To prove this, we show that the black-out level  $s_{K,N}$  as  $N$  gets large is

$$1 - s_{K,N} = \frac{1}{f_B(1)} \frac{a_K}{N} + o(1/N). \quad (\text{A17})$$

Theorem 4.2.3 of Embrechts, Klüppelberg and Mikosch (2012) then implies that

$$\lim_{N \rightarrow \infty} \Pr(Y_{k,N} > s_{K,N} | G) = 1 - e^{-\lambda a_K} \sum_{r=0}^{K-1} \frac{(\lambda a_K)^r}{r!} < 1,$$

which proves that the expected surplus is less than  $\pi X$  since the project is financed only if  $Y_{k,N} > s_{K,N}$ .



Consider therefore the black-out level  $s_{K,N}$ . Notice that by Assumption 2  $s_{K,N} > 0$ . Suppose an investor who submits a bid just above  $s_{K,N}$  is among winning bidders. The most positive signal realization possible is that  $K - 1$  investors get the top signal and the  $K + 1^{th}$  investor receive  $s_{K,N}$  signal. In this case, the likelihood  $z = \frac{\pi}{1-\pi}$  is updated as

$$z\lambda^{K-1} \frac{F_G^{N-K-1}(s_{K,N}) f_G^2(s_{K,N})}{F_B^{N-K-1}(s_{K,N}) f_B^2(s_{K,N})}.$$

Hence, the level of  $s_{K,N}$  that makes the project break-even is

$$z\lambda^{K-1} X \frac{F_G^{N-K-1}(s_{K,N}) f_G^2(s_{K,N})}{F_B^{N-K-1}(s_{K,N}) f_B^2(s_{K,N})} = 1. \quad (\text{A18})$$

Condition (A18) is similar to condition (A7). Following similar steps as in the proof of Proposition 3 we obtain that

$$1 - s_{K,N} = \frac{1}{f_B(1)} \frac{a_K}{N} + o(1/N), \quad a_K = \frac{\ln(\lambda^{K+1} z X)}{\lambda - 1}. \quad (\text{A19})$$

Next, we prove that if  $k/N \rightarrow (1 - \alpha)$ ,  $\alpha \in (0, 1)$  as  $N \rightarrow \infty$  then the expected surplus even in the least efficient equilibrium converges to  $\pi X$ . Thus, we assume that the bids are not revealed after the auction and that the the decision to start the project lies with the  $K^{th}$  highest bidder. The highest black-out level possible is

$$\Pr(G|Y_{K,N} = s_{K,N}^*) (1 + X) - 1 \leq 0. \quad (\text{A20})$$

Equation (A20) implies that

$$\frac{\pi X}{1 - \pi} \frac{F_G^{N-k}(s_{k,N}^*) (1 - F_G(s_{k,N}^*))^{k-1} f_G(s_{k,N}^*)}{F_B^{N-k}(s_{k,N}^*) (1 - F_B(s_{k,N}^*))^{k-1} f_B(s_{k,N}^*)} = 1. \quad (\text{A21})$$

The project is started whenever  $Y_{k,N} > s_{k,N}^*$ . If  $k/N = 1 - \alpha$  then we can write equation (A21) as

$$\frac{\pi X}{1 - \pi} \left( \frac{F_G(s_{k,N}^*)^\alpha (1 - F_G(s_{k,N}^*))^{1-\alpha}}{F_B(s_{k,N}^*)^\alpha (1 - F_B(s_{k,N}^*))^{1-\alpha}} \right)^N \frac{(1 - F_B(s_{k,N}^*)) f_G(s_{k,N}^*)}{(1 - F_G(s_{k,N}^*)) f_B(s_{k,N}^*)} = 1.$$

As  $N$  goes to infinity  $s_{k,N}^*$  converges to the value  $s_k^*$ , which solves

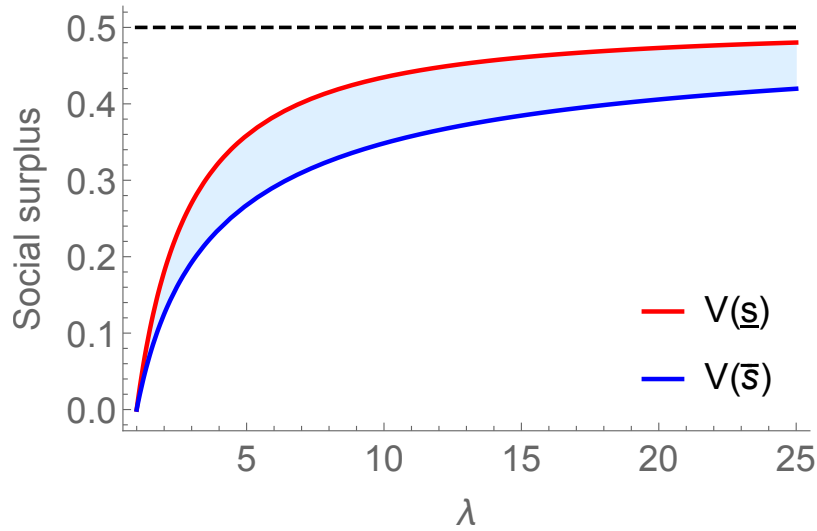
$$F_G(s_k^*)^\alpha (1 - F_G(s_k^*))^{1-\alpha} = F_B(s_k^*)^\alpha (1 - F_B(s_k^*))^{1-\alpha}. \quad (\text{A22})$$

Let  $x_{\alpha,G}$  and  $x_{\alpha,B}$  be such that  $F_G(x_{\alpha,G}) = \alpha$  and  $F_B(x_{\alpha,B}) = \alpha$ . Because of the MLRP  $x_{\alpha,B} < x_{\alpha,G}$ . Notice that  $x^\alpha(1-x)^{1-\alpha}$  is a single-picked function that reaches its maximum at  $x = \alpha$ . Therefore,  $x_{\alpha,B} < s_k^* < x_{\alpha,G}$ .

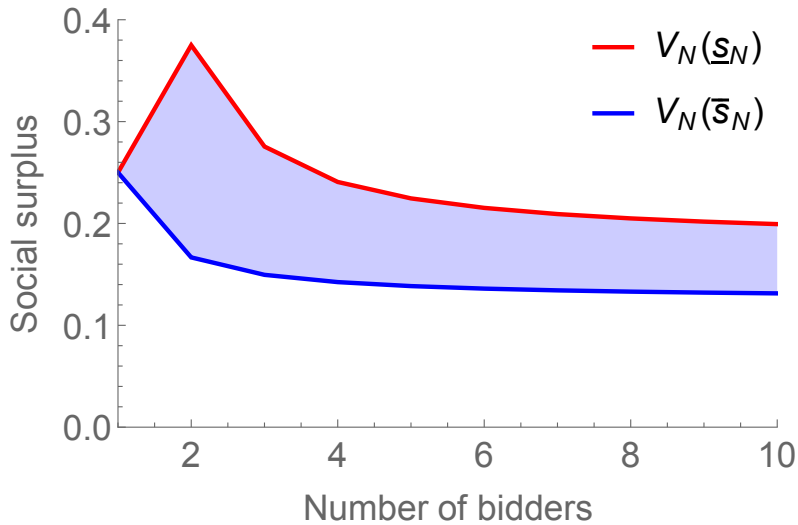
As  $N \rightarrow \infty$  and  $k/N \rightarrow 1 - \alpha$ ,  $Y_{k,N}$  becomes an  $\alpha^{th}$  sample quantile. It is well-known that

$$\sqrt{N}(Y_{k,N} - x_\alpha) \xrightarrow{d} N(0, \alpha(1-\alpha)/f(x_\alpha)^2),$$

where  $f(x)$  and  $F(x)$  are pdf and cdf of observations and  $F(x_\alpha) = \alpha$ . Hence, as  $N \rightarrow \infty$  the probability of undertaking the project goes to one if the project is good and goes to zero if the project is bad. *Q.E.D.*



**Figure 1. Informativeness of top signals and social surplus.** Figure 1 plots limiting social surplus as a function of  $\lambda = f_G(1)/f_B(1)$  when  $N$  grows without bounds. It is assumed that  $\pi = 1/2$  and  $X = 1$ . The red (blue) line corresponds to the most (least) efficient equilibrium. The dashed line shows the first-best surplus  $\pi X$  if information from all bidders is perfectly aggregated.



**Figure 2. Number of bidders and social surplus.** Figure 2 plots social surplus as a function of number of bidders in the setting of Example 1 with  $q_B = 1/2$  and  $q_G = 1$ . The red (blue) line corresponds to the most (least) efficient equilibrium.